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BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

THE TWENTY-EIGHTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

THE twenty-eighth summer meeting of the Society was held at Wellesley College, Wellesley, Massachusetts, September 7–9, 1921, in conjunction with the meeting of the Mathematical Association of America. Joint sessions devoted to discussions of the theory of relativity were held by the two organizations on Wednesday afternoon and Thursday morning (September 7–8). The joint dinner on Wednesday evening was attended by one hundred eleven members and friends. At the close of the sessions, a resolution was passed thanking the college authorities and the Wellesley members of the committee on arrangements for their generous hospitality.

Tower Court was opened for the accommodation of the visitors. By courtesy of Mrs. Walter Hunnewell, the Hunnewell Gardens were open to visitors on Tuesday afternoon. On Tuesday evening an organ recital was given in Houghton Memorial Chapel by Professor H. C. Macdougall of Wellesley College, after which many embraced the opportunity of visiting Whitin Observatory. On Wednesday afternoon, the Denton Brothers' remarkable collection of butterflies was an attraction to many. On Thursday evening, the treasure room of the Library was open, and many ancient books and manuscripts, both mathematical and otherwise, were on exhibition. An automobile excursion to Concord and Lexington was held on Friday afternoon.

Besides the joint sessions, there were regular meetings for the presentation of papers on Thursday afternoon and on Friday morning. The attendance at all the scientific sessions included the following ninety-one members of the Society:

Archibald, Bacon, Barnett, Barney, S. R. Benedict, Bennett, Bill, Birkhoff, Blair, Bliss, Blodgett, Borden, E. W. Brown,

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Cairns, B. H. Camp, Chace, G. M. Conwell, Coolidge, Copeland, Crum, Currier, Dimick, Doak, Donohue, Durfee, Eiesland, Eisenhart, Ettlinger, Falconer, Foberg, Fry, Gillespie, Gilman, Glenn, Gummer, Haskins, Hedrick, Archibald Henderson, Horton, Howland, Hulburt, Huntington, Hurwitz, Dunham Jackson, Kellogg, W. D. Lambert, F. P. Lewis, Longley, Lunn, Mathews, H. A. Merrill, A. L. Miller, G. A. Miller, H. H. Mitchell, C. N. Moore, H. C. M. Morse, Musselman, Norwood, Osgood, Pàlmié, Pell, Pierpont, V. C. Poor, Rambo, C. N. Reynolds, Harris Rice, L. H. Rice, R. G. D. Richardson, Ritt, E. D. Roe, J. R. Roe, Rutledge, Safford, Simon, C. E. Smith, P. F. Smith, Tripp, Tyler, Vivian, J. L. Walsh, Webster, H. S. White, Whittemore, Wiener, R. G. Wood, F. S. Woods, Worthington, Yeaton, J. M. Young, J. W. Young, Zeldin.

President Bliss occupied the chair, being relieved by Vice-President Jackson and by Professor G. A. Miller.

The Council met in Tower Court on Thursday. The following were elected to membership in the Society:

Dr. Nina May Alderton, University of California;
Professor James Atkins Bullard, United States Naval Academy;
Professor Frank Hollinger Clutz, Pennsylvania College;
Dr. Paul H. Daus, University of California;
Mr. James Strode Elston, Travelers Insurance Company;
Mr. Laurence Monroe Klauber, San Diego Consolidated Gas and Electric Company;
Dr. Cyrus Colton MacDuffee, Princeton University;
Mr. Edward Bontecou Morris, Travelers Insurance Company;
Professor Henry Martyn Robert, United States Naval Academy;
President Levi Stephen Shively, Mount Morris College;
Dr. Daniel Victor Steed, University of California.

Thirty applications for membership were received.

The report of the Budget Committee indicated a deficit for the year 1921 and a larger one in prospect for 1922, unless prompt measures are taken to remedy the situation. While there has been an increase in receipts over 1916 of about four thousand dollars, the expenditures have mounted more than five thousand, due in large measure to the increase in the cost of printing. It was decided to inaugurate a campaign for new members, and to this end to enlist the interest and assistance of all members of the Society. In order to facilitate this campaign, By-Law I was suspended for the months November, 1921, to February, 1922, inclusive.

It was voted that the Mathematical Society of France be approached in regard to a reciprocity agreement concerning dues. Committees were appointed to deal with the following matters: life membership fees and funds; the desirability of publishing the Chicago Colloquium lectures; sale and acquisition of back numbers of the journals and other publications of the Society; arrangements for the annual meeting.

The following papers were read at the joint sessions with the Mathematical Association of America:

Some mathematical aspects of the theory of relativity, by Professor James Pierpont.

The place of the Einstein theory in theoretical physics, by Professor A. C. Lunn.

Titles and abstracts of the papers read at the regular sessions follow below. Professor Schwatt's first two papers were read by Professor Eisenhart and his third paper by Professor H. H. Mitchell. The papers of Professor Kasner, Miss Hausle, Professors Synge, Fischer, Kline, R. L. Moore, and Dodd, Dr. Gronwall, Professor Alexander, Professor Rowe, Dr. Wiener, Dr. Zeldin, and Professor Porter, were read by title.

1. Professor L. P. Eisenhart: *Einstein static fields which admit a continuous group G_2 of transformations into themselves.*

For static phenomena in the Einstein theory the linear element of the space-time continuum can be taken in the form $ds^2 = V^2dx_0^2 - ds_0^2$, where $ds_0^2 = \Sigma a_{ik}dx_idx_k$ for $i, k = 1, 2, 3$ is the linear element of the physical space S; and V_0 and the functions a_{ik} $(= a_{ki})$ are independent of x_0, the coordinate of time. Bianchi has found that there are two and only two types of three-spaces S that admit a continuous group G_2 of transformations into themselves. The author proposed the problem of finding the physical spaces, referred to above, which admit such transformations, with the idea that the question of symmetry of an Einstein space should be looked upon in this manner. The problem has been solved in this paper with the result that there are two classes of spaces of the first type of Bianchi, and four of the second type. In all cases the quantities a_{ik} are expressible as algebraic functions of one or two of the coordinates x_i $(i \neq 0)$. The spaces found are, in general, different from any previously known, as is shown by their geometric properties.

2. Professor John Eiesland: *On the class of a certain type of Einstein spreads.*

The author investigates the class of an Einstein spread having the following line element:

$$-ds^2 = (1+\varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi_1 dt^2, \tag{1}$$

in which R, θ, φ are spherical coordinates and φ_1, φ_2 are arbitrary functions of R. The following theorems are proved:

1. If the space (1) is of the first class, the condition

$$\varphi_1'' = \frac{1}{2}\left[\frac{\varphi_1'\varphi_2'}{\varphi_2} + \frac{(\varphi_1')^2}{\varphi_1}\right] \tag{2}$$

must be satisfied, and conversely.

2. A space (1) is at most of the second class.

A representation of the space in a flat 5-space is obtained for the case that (2) is satisfied, and another representation in a flat 6-space when (2) is not satisfied. With a slight modification of the condition (2), these theorems also hold for the more general stationary space

$$-ds^2 = (1+\varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi_3 dR dt - \varphi_1 dt^2.$$

Applications are made to various spaces in the general theory.

3. Professor Edward Kasner: *The solar gravitational field and certain other fields completely determined by light rays.*

In this paper, which has been submitted for publication to the MATHEMATISCHE ANNALEN, it is shown that the solar field, taken in the Schwarzschild form, is exactly determined by its light rays.

In a paper (AMERICAN JOURNAL, vol. 43, Jan., 1921, p. 20) the proof was given only for approximately euclidean forms. The present proof is valid for the complete equations. The same general method is then applied to the Minkowski form.

Finally, the discussion is carried out for a very simple class of Einstein fields, namely, those in which the coefficients of the squares of the differentials are all functions of one variable, say x_1 (see this BULLETIN, vol. 27, p. 62). It is shown that such fields are exactly determined by their light properties, and therefore the orbits (geodesics) can be calculated if we know merely the light rays (nullgeodesics).

These fields may be represented as four-spreads imbedded in a seven-flat, in contrast to the solar field which the author has shown can be imbedded in a six-flat (AMERICAN JOURNAL, vol. 43, April, 1921, p. 130).

A corresponding discussion of dimensionality is carried out for Einstein's more general cosmological equations

$$G_{ik} - \tfrac{1}{4}g_{ik}G = 0.$$

4. Professor G. A. Miller: *Prime-power groups containing only one invariant subgroup of every index which exceeds this prime number.*

Among the theorems proved are the following: If a non-cyclic group of order p^m, p being a prime number, contains only one invariant subgroup of every index which exceeds p, it must contain a cyclic subgroup of index p when $p = 2$, but when $p > 2$, each of its invariant subgroups whose order exceeds p must be non-cyclic. In all cases such a group must contain a subgroup of index p which includes all its operators whose orders exceed p^2. When $m > p + 1$ there are three and only three non-cyclic groups of order p^m which have the property that each of them contains only one invariant subgroup of every index which exceeds p and one abelian subgroup of index p. If H_α is such an invariant subgroup of order p^α then the pth power of every operator in H_α, $m > \alpha > 0$, is found in $H_{\alpha-p+1}$, where $\alpha - p + 1$ is to be replaced by zero whenever it is a negative number or zero.

5. Professor G. D. Birkhoff: *General mean-value relations.*

In this paper the following general problem is treated: a function $f(x)$ and its derivatives are connected by one or more polynomial equations in which x has values $x_1, x_2, \cdots, x_n$; to determine when a specified linear differential expression in f necessarily vanishes within the interval of the x_i's.

6. Professor G. D. Birkhoff: *On plates of variable thickness.*

This paper contains a general method for the development of the elastic properties of plates of variable thickness. Only the case of constant thickness has been hitherto considered.

7. Professor E. V. Huntington: *Application of least squares to the problem of apportionment.*

The problem of apportionment may be attacked in two ways: (1) by the comparison plan, which seeks to minimize the inequality between each state and every other state; and

(2) by the summation plan, which seeks to minimize the total error of the apportionment. The comparison plan, as shown by the writer in earlier papers (presented December, 1920, and February, 1921), leads to only five distinct methods, among which the choice of the *method of equal proportions* appears to be clearly indicated. (See AMERICAN STATISTICAL QUARTERLY, Sept. 1921.) The summation plan, on the other hand, which was used by Professor Owens in his papers at the February and April meetings, is here shown to be unsatisfactory, since it leads to so many different methods that it would be difficult to make a convincing choice between them. It may be noted that the only methods which appear under both plans are the method of equal proportions and the Willcox-Owens method of major fractions. The paper concludes with a *rank-list* of all the seventy-two known methods arranged in the order in which they favor the larger states. In this list, the method of equal proportions occupies, in a genuine sense, the middle position.

8. Professor I. J. Schwatt: *The summation of series by means of generating functions.*

This paper gives a method for finding the sums of certain series, such as

$$\sum_{k=0}^{n}\binom{2n-k}{k}m^k, \qquad \sum_{k=0}^{n}k^p, \qquad \sum_{k=0}^{[p/2]}\binom{n}{k}\binom{n-k}{p-2k}2^{p-2k},$$

which the author was unable to obtain by any other method.

9. Professor I. J. Schwatt: *The expansion of any power of a multinomial.*

The author obtains a series expansion, $\Sigma_{n=0}^{\infty}(x^n/n!)P_n$, for any power of a polynomial, $y = (\Sigma_{m=0}^{h}a_m x^m)^p$, where p is any real number. The coefficients P_n are numbers easily obtained from an expression for the nth derivative of y. The special case of p a positive integer is also considered.

10. Professor I. J. Schwatt: *The operator* $[r(d/dr)]$ *on* $F(r)$.

It is shown that for any analytic function $F(r)$

$$\left(r\frac{d}{dr}\right)^p F(r) = \sum_{n=1}^{p}\frac{(-1)^n}{n!}\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^p r^n\frac{d^n}{dr^n}F(r).$$

Several applications are given.

11. Miss Eugenie C. Hausle (introduced by Professor Edward Kasner): *Geometric characterization of special singly infinite families of heat curves.*

The author has considered the geometric characterization of the singly infinite families of curves derived from certain limitations on the Fourier equation of heat. It is known that a system of curves $\varphi(x, y, t) = T$, where t denotes the time and T the temperature, obeys the Fourier equation. If the heat is required to be in equilibrium, $\partial\phi/\partial t = 0$. Let this be Case I, the isothermal case. Case II is obtained by putting $\partial\phi/\partial t = 1$ and Case III by putting $\partial\phi/\partial t = \varphi$.

Consider Case I. The equation of the given family of curves $\varphi(x, y) = c$ and Laplace's equation must be satisfied simultaneously. Then $f = dy/dx$ satisfies Laplace's equation. If γ and $\bar{\gamma}$ denote the curvatures of the given curves and their orthogonal trajectories, the preceding result gives (1) $\gamma_s + \bar{\gamma}_{\bar{s}} = 0$ as the geometric property, where $\gamma_s = dr/ds$.

In Case II, two relations are obtained, each of the fourth order. In Case III, two relations are obtained, one of which is identical with one of those in Case II. These relations completely characterize all ∞^1 systems of heat curves.

12. Professor J. L. Synge: *On the stability of a bicycle with a light frame.*

The bicycle under discussion consists of two similar wheels —solids of revolution with equatorial planes of symmetry— and two weightless frame pieces. The stability is discussed in the case in which the system has uniform rectilinear motion, the wheels rolling on a rough horizontal plane. Use is made of Routh's conditions that no zero of a polynomial should have its real part positive. The conditions for stability differ with the detail of the specification, and five results are given. Of these, two are of especial interest, viz. (1) that for certain specifications the motion is stable for speeds up to, but not exceeding, a certain limit; (2) that when the common axis of the frame pieces is vertical in steady motion, there is instability at all speeds. Definite examples are given.

13. Professor C. A. Fischer: *Note on the definition of a linear functional.*

A linear functional has usually been defined as one which is distributive and continuous, but a continuous functional has been defined in at least two ways which are not equivalent.

F. Riesz, Fréchet and others have defined a continuous functional as one which satisfies the equation

$$\lim_{n\to\infty} L(u_n(x)) = L(u(x)),$$

when the sequence of u_n's approaches $u(x)$ uniformly. More recently Lévy and W. L. Hart have required only that it converge to $u(x)$ in the mean. F. Riesz has proved that if L is distributive and the above equation is satisfied for uniformly convergent sequences, the functional is equivalent to a Stieltjes integral, such as $\int_a^b u(x)d\alpha(x)$.

In the present note it is proved that if the same equation is satisfied when the u_n's are required only to converge in the mean, the functional is equivalent to a Lebesgue integral such as $\int_a^b u(x)\beta(x)dx$.

14. Professor J. R. Kline: *Certain theorems concerning simple closed and open curves.*

In order that a closed connected set M should be a simple closed curve, the author shows that it is necessary and sufficient that M remain connected upon the removal of any proper connected subset. Thus there is no unbounded closed connected set M which is such that $M - g$ is connected for every proper connected subset g of M. However, if M is an unbounded closed connected set that remains connected upon the removal of any *unbounded* proper connected subset, then M must be either an open curve, a ray of an open curve, or the point set composed of a simple closed curve J and a ray of an open curve starting from P such that the ray has P and only P in common with J.

15. Professor J. R. Kline: *A theorem concerning connected sets which become totally disconnected upon the removal of a single point.*

An example was recently given in the FUNDAMENTA MATHEMATICÆ of a connected point set M which contains a point P such that $M - P$ is totally disconnected, i.e. $M - P$ contains no connected subset consisting of more than a single point. Professor Kline shows that no connected set M can have more than one point P such that $M - P$ is totally disconnected. Indeed, if a connected set M has one point P such that $M - P$ is totally disconnected, then, if Q is any point of M different from P, $M - Q$ is connected.

16. Professor R. L. Moore: *Concerning connectedness im kleinen and a related property.*

Sierpinski has recently* shown that in order that a closed connected point set M should be a continuous curve it is necessary and sufficient that, for every positive number ϵ, M should be the sum of a finite number of *closed*, connected point sets each of diameter less than ϵ.

A point set will be said to have property S if, for every positive number ϵ, it is the sum of a finite number of connected point sets, each of diameter less than ϵ. Professor Moore shows that, as applied to bounded connected point sets, condition S is stronger than that of connectedness im kleinen† but weaker than that of *uniform* connectedness im kleinen. He shows also that in order that a simply connected bounded domain should have a continuous curve for its boundary, it is necessary and sufficient that it should have property S. Finally, it is also shown that if a bounded point set M is uniformly connected im kleinen, then the point set composed of M plus its boundary is connected im kleinen.

17. Professor E. L. Dodd: *The probability function for the sum of certain functions, with applications to the theory of errors.*

This paper extends a theorem of von Mises,‡ who, by the use of the Stieltjes integral, treats together continuous and discontinuous probability. The theorem for the sum of variables is extended to the sum of functions, each limited in a finite interval, and having a limited number of oscillations therein, and with moments absolutely convergent.

The general mean of n measurements m_k is defined as $g[\Sigma c_k f_k(m_k) \div \Sigma c_k]$ where the c_k are positive constants, the f_k are continuous increasing functions, and g is the inverse of $\Sigma c_k f_k(\xi) \div \Sigma c_k$. If each $f_k = f$, and $0 < c \leqq c_k \leqq d$, and the errors are small in comparison with the measurements, the precision of this mean increases as the square root of n; but a constant error will appear that is sometimes negligible.

18. Dr. T. H. Gronwall: *On power series with positive real part in the unit circle.*

Carathéodory has given the necessary and sufficient conditions on the n constants $a_1, a_2, \cdots, a_n$ in order that there shall exist a $\varphi(z)$, holomorphic and of positive real part for $|z| < 1$, the expansion of which begins with

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* FUNDAMENTA MATHEMATICÆ, vol. 1 (1920), pp. 44–60.

† Cf. H. Hahn.

‡ *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, MATHEMATISCHE ZEITSCHRIFT, vol. 4, pp. 24, 54, 61, 80, 81.

$$\varphi(z) = \tfrac{1}{2} + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots.$$

Using the methods of Minkowski's geometry of convex solids, he expresses the a's in parametric form, and through the algebra of Hermitian forms, Toeplitz and Fischer have transformed these conditions into algebraic inequalities involving $a_1, a_2, \cdots, a_n$ and their conjugates. In the present paper the results of these authors are proved by the most elementary function theoretic means, consisting mainly in the combination of complete induction with Schwarz's lemma.

19. Professor J. W. Alexander: *Some theorems on transformations with invariant points.*

This paper gives a new proof of an extended form of Brouwer's theorem on the transformation of an n-sphere into itself, and a number of applications of the generalized theorem.

20. Professor J. W. Alexander: *Theorem on the interior of a simply connected closed surface in three-space.*

In this paper it is proved that if S be a simply connected closed surface in three-space the interior and boundary of S may be mapped uniformly on the interior and boundary of a sphere. The proof can be generalized to n dimensions.

21. Dr. H. M. Morse: *A fundamental class of geodesics on closed surfaces of genus greater than unity.*

Dr. Morse in this paper considers geodesics, termed geodesics of class A, every segment of which, however extended, is at least as short as any other curve joining the end points of the given segment. The fundamental theorem of this paper is that corresponding to any semicircle lying in the upper half plane and with end points on the axis of reals, there exists on the given surface a geodesic of class A possessing an image on the half plane which if extended indefinitely in either sense approaches respectively the two end points of the semicircle, and such that the semicircle and image of the geodesic can be traced out simultaneously by two moving points, capable of being joined by a continuously moving curve whose image on the given surface remains less in length than some fixed constant. Conversely, corresponding to every geodesic of class A there is associated in the above manner some semicircle with end points on the axis of reals.

22. Professor A. G. Webster: *On the problem of steering an automobile around a corner.*

In order to avoid skidding, there must be an upper limit to the centrifugal force. If we seek to make the mean square of this a minimum, we find that the curve should be the elastica. The kinematics of steering is examined, as are the results of the hypothesis that in driving at constant speed the wheel is put over at constant velocity. The intrinsic equation of the curve is found, and in practical cases it, like the elastica, differs little from the Cornu spiral used in optics. In practice all these curves (that is for the first half, which is symmetrically repeated) may be well represented by a cubical parabola. Drawings are given, and safe speeds are stated.

23. Professor A. G. Webster: *On the principles of mechanical integrators for differential equations, especially those of exterior ballistics.*

The usual forms of integrator involve the revolving plate and wheel at various distances from the center, or the steered wheel which rolls in the direction of the integrated curve. To this the writer proposes to add a third device, like an automobile with the front wheels at right angles to the wheelbase, the length of which varies. Such a device enables one to draw involutes. In the proposed machine, two integrators are combined, one of which rides on top of the other. One describes the hodograph, the other the evolute of the trajectory and the trajectory itself. Templets or cams insert once for all the law of resistance of the air and the law of density-change with altitude. The trajectory is drawn at one operation, although the equations are of the second order.

24. Professor C. N. Moore: *On the Fourier's series of non-integrable functions.*

An integral over a finite range which fails to converge on account of the nature of the infinite discontinuities of the integrand may be summable by methods analogous to those used for divergent series and divergent integrals over an infinite range. In this way Fourier's series may be obtained which correspond to functions that are non-integrable in the ordinary sense. The principal result of this paper is the theorem that if the integral of a function is summable (Cr) for any $r > 0$, the corresponding Fourier's series will be summable (C1) to the value $\lim_{h\to 0}\,[\frac{1}{2}\{f(x+h)+f(x-h)\}]$ at every point for which this limit exists.

25. Professor C. F. Gummer: *A generalization of Laguerre's rule of signs.*

It was shown by Laguerre that, if $c_1, c_2, \cdots, c_n$ are any real numbers and if $p_1, p_2, \cdots, p_n$ are a descending sequence of rational numbers, the sum of the multiplicities of those roots of the equation $\Sigma_i c_i x^{p_i} = 0$ that exceed unity is not greater than the number of variations of sign in the following sequence: $c_1, c_1 + c_2, c_1 + c_2 + c_3, \cdots, c_1 + \cdots + c_n$. In this paper a proof is given that allows the p's to be irrational. The theorem is applied to some special types of equation.

26. Professor R. E. Gilman: *The functions analogous to Lebesgue constants for a series of Hermite polynomials.*

To each member of the class of functions absolutely integrable in the interval $-\infty$ to $+\infty$ and such that $|f(x)| \leqq 1$ there corresponds a formal expansion in Hermite polynomials, which may or may not be convergent. If the sum of the first n terms of this expansion be denoted by $s(x, n)$, there is a function of the given class which, for a fixed n and for a fixed $x = x_0$ on the given interval, makes $s(x_0, n)$ a maximum. The author determines the order of magnitude with respect to n of ρ_n, where ρ_n is the maximum of $s(x_0, n)$.

27. Professor O. E. Glenn: *Theory of invariant elements.*

Many theories tend toward the unity exemplified in general analysis. The theory of invariant elements, which is the foundation of the methods of the present paper, has furnished the fundamental viewpoint in papers by the writer on algebraic, modular, differential, and special algebraic invariant theories, particularly for binary quantics. The theory here developed is a synthesis based upon these with extension to n variables under a form as thoroughly definitive as seems feasible.

28. Dr. J. L. Walsh: *On the location of the roots of the jacobian of two binary forms.*

This paper presents the following result: Let $g(z)$ denote a polynomial whose roots of respective multiplicities $m_1, m_2, \cdots, m_n$ lie at the collinear points $\alpha_1, \alpha_2, \cdots, \alpha_n$, and let $\alpha_1', \alpha_2', \cdots, \alpha_n'$ denote the roots, whose multiplicities are $m_1', m_2', \cdots, m_n'$, of the derivative $g'(z)$. Suppose the loci of m_i roots of a polynomial $f(z)$ to be the interior and boundary of a circle C_i whose center is α_i and radius r $(i = 1, 2, \cdots, n)$, and that $f(z)$ has no other roots. Then the loci of the roots of $f'(z)$ are the interiors and boundaries of circles C_i' of centers $\alpha_1', \alpha_2', \cdots, \alpha_n'$ and common radius r. A circle C_i' exterior to all other circles C_j' contains m_i' roots of $f'(z)$.

This result can be extended to a number of circles C_i that have a common external center of similitude, to the derivatives of $f(z)$ of all orders, and to the jacobian of two binary forms.

29. Professor J. E. Rowe: *The power of a modern gun and of thunder.*

The first part of the paper is devoted to the comparison of the power at which a large gun works during the time of actual performance, and the power of a man. The second part is devoted to the explanation of a method by which the magnitude of the air disturbance in thunder may be calculated.

30. Professor J. R. Musselman: *Spurious correlation applied to urn schemata.*

In games of pure chance such as balls drawn from an urn, or coin-tossing, or dice-throwing some scheme must always be introduced to obtain correlation between the two sets of drawings, and hence the correlation arising is "spurious" correlation. The author applies the idea of spurious correlation to various urn schemata and obtains some interesting theorems. This method gives a simple proof for that urn schema of Rietz which has t/s for its correlation coefficient.

31. Professor W. L. Crum: *The significance of the partial correlation coefficient in the comparison of ordered statistical series possessing rectilinear trends.*

It is well known that, if two ordered statistical series, for instance two historical economic series, are subject to rectilinear trends, their correlation coefficient is influenced by that fact. In order to determine the degree of correlation which is independent of the normal trends, it is customary to *correct* the original items by subtracting the corresponding ordinates of their respective lines of trend. This paper shows that, when there is no lag in the correlation, the coefficient of correlation between the residuals obtained by the above process of correction is precisely the partial correlation coefficient between the two given series. In case there is lag, it is shown that the partial correlation coefficient must be altered by the introduction of a multiplier which is the correlation between the two time series, one of which is displaced relative to the other by an amount equal to the lag.

32. Professor W. L. Crum: *A tentative substitute for the standard deviation in the examination of the dispersion of an ordered statistical series.*

A statistical series may be assigned to one of two broad classes according as it consists merely of a list of numbers of independent arrangement, or has its items ordered relative to a particular variable. Typical of the first class are the series composed of experimental measurements, and the chief illustrations of the second class are to be found in the historical series in the field of economic statistics. The paper points out that the standard deviation fails to give full information about the dispersion of series of the second class, and, in particular, takes no account of a phase of the dispersion which may be called the rate of fluctuation. With a view to bringing out certain facts about this rate, a study is made of the coefficient of correlation between the items of the given series and the values of the variable relative to which these items are ordered, and it is shown that this coefficient is not an adequate measure of the rate. Finally the square root of the mean squared second order difference is set up tentatively as a general measure of the fluctuation-rate for ordered statistical series.

33. Professor B. H. Camp: *The value of a sample. Second paper.*

The fundamental theorem of the paper is this: Let $f(x)$ be any frequency distribution whatever, and let y be equal to the mean of k x's drawn, with replacements, from $f(x)$. The rth moment of the distribution of the y's approaches as a limit the value of the rth moment of the Gaussian curve as k becomes infinite.

The more important applications of this theorem center round a method of approximating the difference between the rth moment and its limit in special cases. This enables one to use advantageously a generalization of Tchebycheff's criterion, discovered first by Pearson, for finding the probability that a random mean will differ from its ideal value by as much as a prescribed amount.

Taken in conjunction with another theorem announced in the author's earlier paper, it furnishes a method of obtaining high moments of a point binomial. These are valuable in the theory of simple sampling, but the labor of computing them has been prohibitive.

34. Dr. Norbert Wiener: *A form of series for potential problems.*

This paper develops a generalization of the notion of sets of normal and orthogonal functions by substituting the rule of combination

$$f(x, y, z) \mid g(x, y, z) = \iiint_V \Delta f \Delta g dr + \iint_S fg d\sigma,$$

where V is a region of three-space and S its boundary, for the rule of combination $\int_a^b u(x)v(x)dx$ used in the definition of ordinary orthogonality. Application of these generalized normalized sets is made to the solution of Dirichlet's problem and similar problems concerning Poisson's equation.

35. Dr. S. D. Zeldin: *Some hydrodynamic aspects of group theory.*

It is known that the steady motion of a fluid can be represented by a one-parameter continuous group in three variables. By finding the functions f which admit the infinitesimal transformation

$$Kf = u(x, y, z)\frac{\partial f}{\partial x} + v(x, y, z)\frac{\partial f}{\partial y} + w(x, y, z)\frac{\partial f}{\partial z},$$

the lines of flow can be determined. This paper deals with the invariant configurations which arise by finding the points satisfying equations $u = 0, v = 0, w = 0$. The groups particularly considered are of projective and linearoid types.

36. Professor M. B. Porter: *Two-way series for Lebesgue integrals.*

The author proposes to define, *ab initio*, the integral of any positive summable function in n variables by a two-way series. Most of the essential theorems of Lebesgue's theory then follow from the density theorem, which is an immediate corollary of Vitali's covering theorem. In case the integrand is not of one sign, the form of the series is altered. This leads to the definition of integrals for *non*-summable functions, among others Harnack integrals and Cauchy principal-value integrals.

This paper will appear in a later number of this BULLETIN.

R. G. D. RICHARDSON,

Secretary.

A LETTER FROM THE PRESIDENT

To the Members of the American Mathematical Society:

In the autumn of 1920 a committee under the leadership of Professor E. R. Hedrick inaugurated a campaign to increase the membership of the Society and to secure new subscriptions to our journals. The effort was successful, but we have been surprised to find that over sixty per cent of the mathematicians in our colleges and universities are still not members.

The campaign was undertaken on account of the greatly increased costs of administration, and especially of printing, which have followed the great war. By securing special funds which will not be available in future years the sizes of the Bulletin and Transactions have been kept nearly normal during 1921, but the Society is facing the possibility of serious reductions in 1922 if our resources can not be increased.

For this reason the Council of the Society has authorized the continuation of the campaign for memberships and subscriptions during the years 1921 and 1922. Many of the members of the Society have taken effective interest in the campaign, and their assistance has been greatly appreciated by the membership committee and the officers of the Society. It is possible that others also could help materially if they understood the importance of such assistance during this critical period in the development of our American mathematical school.

I am taking this occasion, therefore, to remind our members of the needs of the Society at the present time, and to urge them to seek opportunities to increase the sizes of our membership and subscription lists. This can be done, for example, by making sure that colleagues who are not members have reasonably considered the desirability of membership in the Society, by subscribing to the Transactions, or by securing subscriptions to the Bulletin and Transactions from your college and city libraries.*

GILBERT A. BLISS,
President of the American Mathematical Society.

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* See the announcement of special rates for back numbers of our journals on p. iii near the back cover of this number.

REPORT ON
TOPICS IN THE THEORY OF DIVERGENT SERIES*

BY W. A. HURWITZ

1. *Introduction.* It is with some reluctance that I have acceded to the request of the programme committee to address the Society on the present status of problems concerning divergent series, since three admirable expository treatments of this field have already been given to the Society and published in the BULLETIN, by W. B. Ford,† R. D. Carmichael,‡ and C. N. Moore.§ In particular Professor Carmichael has so closely followed the trend which my own thoughts have taken (except that I should not have presented them so elegantly) as to make it difficult for me to give an adequate account without a good deal of repetition. At any rate I shall take advantage of his paper in order to plunge *in medias res* today, and also to omit references to original sources unless these seem especially desirable.

If we have the symbol

$$\Sigma u_n = u_1 + u_2 + u_3 + \cdots, \tag{1}$$

we define

$$x_n = u_1 + u_2 + \cdots + u_n, \tag{2}$$

so that

$$u_1 = x_1; \qquad u_n = x_n - x_{n-1}, \quad n > 1. \tag{3}$$

In case

$$\lim_{n \to \infty} x_n$$

exists, we say that the series Σu_n or the sequence (x_n) is convergent, the limit of the sequence being the *value* or *sum* of the series. The importane of this concception lies in the fact that many formal transformations carried out on infinite series as if they are finite sums can be proved correct. Instances arose early in the study of series, however, in which such transformations, though assuredly not correct on the basis of existing theory, nevertheless led in special cases to correct results.* The attempt to justify such results led naturally to generalizations of the notion of the value of a series.

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* Presented before the Society at the Symposium held in New York City, April 23, 1921.

† This BULLETIN, vol. 25 (1918–19), p. 1.

‡ *Ibid.*, vol. 25 (1918–19), p. 97.

§ *Ibid.*, vol. 25 (1918–19), p. 258.

I shall denote methods of definition of value by capital letters $A, B, \cdots$; I shall express the fact that a series or sequence can be evaluated by a method A by saying that it is *summable* A; and the resulting value I shall denote by $A\mathbf{L}(x_n)$.

2. *Linearity and Regularity.* One of the important properties of definitions is that of *linearity*. A definition A is said to be *linear* if

$$A\mathbf{L}(x_n + x_n') = A\mathbf{L}(x_n) + A\mathbf{L}(x_n')$$

whenever $A\mathbf{L}(x_n)$ and $A\mathbf{L}(x_n')$ exist, and

$$A\mathbf{L}(cx_n) = cA\mathbf{L}(x_n)$$

whenever $A\mathbf{L}(x_n)$ exists.

I have stated this property first of all, because it is the only one which is satisfied, so far as I know, by every definition which has ever been proposed as practically useful. Of even wider importance of course, although not satisfied by every useful definition, is the following. A definition A is said to be *regular* in case it evaluates every convergent sequence, giving it the value to which it converges; that is, in case

$$\lim_{n\to\infty} x_n = l$$

implies $A\mathbf{L}(x_n) = l$.

Nearly every definition that has been proposed can be thrown into the following form:

Let a point set T be given in space of any number of dimensions, real or complex, having a limit point t_0 (actual or symbolic†) not belonging to T, and let the functions $a_k(t)$ $[k = 1, 2, \cdots]$ be defined in T; then if the sequence (x_n) is such that

$$(G) \qquad y(t) = \sum_{k=1}^{\infty} a_k(t)x_k$$

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* It is only necessary to allude to the interesting history of the series $1 - 1 + 1 - 1 + \cdots$.

† I.e., having one or more of its coordinates infinite.

converges for each t in T and if

$$\lim_{t \to t_0(T)} y(t) = l,$$

the sequence (x_n) is said to be summable G, and $G\mathbf{L}(x_n) = l$. This definition could of course be expressed in terms of u_n instead of x_n.

An important special case is that in which T consists of the positive real integers and in which $a_k(t)$, which may then be written a_{nk}, is zero for $k > n$. In this case the requirement of convergence of G is met automatically, and we may say:

The sequence (x_n) is summable G', and $G'\mathbf{L}(x_n) = l$ provided

$$\lim_{n \to \infty} y_n = l,$$

where

$$(G') \qquad y_n = \sum_{k=1}^{n} a_{nk} x_k.$$

An especial convenience in dealing with this case is that the transformation G' may be treated by the usual algebraic machinery of linear transformations, since the first n y's depend only on the first n x's. We may speak of sums, numerical multiples, products, and powers of G'. The definition of ordinary convergence is one such case; it is given by the transformation

$$(I) \qquad y_n = x_n.$$

Any transformation of the form G' will have an inverse provided $a_{nn} \neq 0$ for every n.

For both G and G' it is obvious that the condition of linearity is satisfied. Sufficient conditions for the regularity of G were given by Silverman;* they were proved to be also necessary (even in a slightly more general case) by Toeplitz:†

For the definition G' to be regular it is necessary and sufficient that

$$(G'1) \qquad \textit{for each } k, \ \lim_{n \to \infty} a_{nk} = 0;$$

$$(G'2) \qquad \textit{for all } n, \ \sum_{k=1}^{n} |a_{nk}| \textit{ is bounded;}$$

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* Ph.D. thesis, Missouri, 1910; UNIVERSITY OF MISSOURI STUDIES, MATHEMATICS SERIES, vol. 1, no. 1 (1913).

† PRACE MATEMATYCZNO-FIZYCZNE, vol. 22 (1911), p. 113.

$$(G'3) \qquad \lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1.$$

The corresponding theorem for the more general definition G is due to Carmichael* and Hildebrandt;† it has recently been restated by J. Schur:‡

For the definition G to be regular it is necessary and sufficient that

$$(G1) \quad \textit{for each } k, \quad \lim_{t\to t_0(T)} a_k(t) = 0;$$

$$(G2) \quad \textit{for each } t \textit{ in } T, \quad \sum_{k=1}^{\infty} |a_k(t)| \textit{ converges,}$$

$$\textit{and for all } t \textit{ in } T, \quad \sum_{k=1}^{\infty} |a_k(t)| \textit{ is bounded;}$$

$$(G3) \qquad \lim_{t\to t_0(T)} \sum_{k=1}^{\infty} a_k(t) = 1.$$

It should be noted that if we restrict all quantities appearing in these theorems to be real, the theorems remain true, not only as regards sufficiency, but also as regards necessity.

3. *Definitions of Summability.* I shall now give examples of definitions, illustrating each by application to the series

$$(4) \qquad 1 + x + x^2 + x^3 + \cdots.$$

For this series $u_n = x^{n-1}$; hence

$$(5) \qquad x_n = \frac{1 - x^n}{1 - x}, \qquad x \neq 1.$$

As the definitions are all to satisfy the requirement of linearity, we can prove that $A\mathbf{L}(x_n) = 1/(1 - x)$ by showing that $A\mathbf{L}(x^n) = 0$. We shall therefore consider instead of (5)

$$(6) \qquad x_n = x^n.$$

The usual definition of limit gives the desired result when and only when $|x| < 1$. Hölder employed the transformation

$$(M) \qquad y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

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* Loc. cit., p. 118.

† Abstract, this BULLETIN, vol. 24 (1917–18), p. 429. Compare also the statement made by Carmichael in the preceding reference.

‡ JOURNAL FÜR MATHEMATIK, vol. 151 (1920), p. 82.

Here, in the notation of G', $a_{nk} = 1/n$. This definition is obviously regular. It constitutes the first Hölder mean. The further Hölder means are conveniently described as algebraic powers of M: $M^2, M^3, \cdots$.

In order to apply M to (6), we have $x_n = x^n$, and for $x \neq 1$,

$$y_n = \frac{x + x^2 + \cdots + x^n}{n} = \frac{x(1 - x^n)}{n(1 - x)}$$

$$= \frac{x}{n(1 - x)} - \frac{x}{1 - x}\,\frac{x^n}{n}.$$

The first term approaches zero as n becomes infinite, and

$$\lim_{n \to \infty} \left| \frac{x^n}{n} \right| = \begin{cases} 0, & |x| \leqq 1, \\ \infty, & |x| > 1; \end{cases}$$

hence (4) is summable M to the value $1/(1 - x)$ for $|x| \leqq 1$, $x \neq 1$, and is not summable M for $|x| > 1$. I shall defer until later the consideration of the behavior for $x = 1$.*

Hence M evaluates (4) to the value which would be expected, not only at points inside the circle of convergence, but also at all points on the circle other than $x = 1$; and at no points outside the circle. It can readily be shown that for any value of r, M^r accomplishes no more for the series (4).

Cesàro gave the formulas:

$$C_1 = M;$$

$$(C_2) \quad y_n = \frac{nx_1 + (n - 1)x_2 + \cdots + x_n}{n(n + 1)/2};$$

. . . .

$$(C_r) \quad y_n = \frac{n \cdots (n + r - 2)\, x_1 + \cdots + 1 \cdots (r - 1)x_n}{n(n + 1) \cdots (n + r - 1)/r}.$$

. . . .

For every positive integer r, C_r is regular.

Extensions of Cesàro's method were given by Knopp and Chapman. Chapman's formula is

$$(C_r) \qquad y_n = \sum_{k=1}^{n} r \frac{(n - 1)!\Gamma(n + r - k)}{(n - k)!\Gamma(n + r)} x_k,$$

which reduces to Cesàro's form when r is a positive integer,

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[*] See § 6.

but has a meaning when r is fractional, irrational or even complex, provided r is not a negative integer or zero. In a limiting sense, it may be said to have a meaning even for $r = 0$: $C_0 = I$. Applying C_2, for illustration, to (6), we have $x_n = x^n$, for $x \neq 1$,

$$y_n = \frac{nx + (n-1)x^2 + \cdots + x^n}{n(n+1)/2},$$

$$(1-x)^2\, y_n = \frac{nx - (n+1)x^2 + x^{n+2}}{n(n+1)/2}$$

$$= \frac{2x}{n+1} - \frac{2x^2}{n} + \frac{2nx^2}{n+1}\left(\frac{x^n}{n^2}\right).$$

Hence as before $C_2\mathbf{L}(x^n) = 0$ for $|x| \leqq 1$, $x \neq 1$; $C_2\mathbf{L}(x^n)$ does not exist if $|x| > 1$. The same result can be obtained for C_r, in the same fashion if r is a positive integer, and by other considerations in any case when $\mathbf{R}(r) > 0$.

It is seen that no definition of Hölder or Cesàro serves to evaluate (4) outside its region of convergence. A convenient form of definition is given by the exponential mean:*

$$(E_r) \qquad y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!}\,\frac{(r-1)^{n-k}}{r^{n-1}}\,x_k.$$

The successive coefficients are the terms of the binomial expansion of $[(1 - 1/r) + 1/r]^{n-1}$. The definition is regular if and only if r is real and $\geqq 1$. It is readily seen that $E_1 = I$. The name "exponential mean" is suggested because of the interesting algebraic property $E_rE_s = E_{rs}$.

Applying E_r to (6), we have

$$y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!}\,\frac{(r-1)^{n-k}}{r^{n-1}}\,x^k$$

$$= x\left[1 - \frac{1}{r} + \frac{x}{r}\right]^{n-1}.$$

Thus y_n will approach zero if and only if

$$\left|1 - \frac{1}{r} + \frac{x}{r}\right| < 1,$$

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* This formula was given, in slightly different notation, by Hausdorff, MATHEMATISCHE ZEITSCHRIFT, vol. 9 (1921), p. 86; his work was unknown to me during the preparation of this paper for presentation.

that is,

$$|x - (1 - r)| < r.$$

The region of evaluation is therefore the interior of a circle through $x = 1$, having its center at $x = 1 - r$. For any $r > 1$, this region includes points outside the circle of convergence. For large enough r any point x will be obtained which satisfies the condition $\mathbf{R}(x) < 1$.

A variety of definitions were given by M. Riesz. Let (λ_n) be any sequence of positive numbers increasing and becoming infinite. For one form of the Riesz mean of type (λ) and order r, we write (in terms of u_n rather than x_n)

$$(R_{\lambda, r}) \qquad y_n = \sum_{k=1}^{n} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^r u_k.$$

For the case $\lambda_n = n$, $r = 1$, this has the form

$$y_n = u_1\left(1 - \frac{1}{n}\right) + u_2\left(1 - \frac{2}{n}\right) + \cdots + u_{n-1}\left(1 - \frac{n-1}{n}\right)$$

$$= \frac{1}{n}(x_1 + \cdots + x_{n-1}),$$

and is essentially the same as M. For any (λ_n), $R_{\lambda, 0} = I$. For all cases in which $r \geqq 0$, $R_{\lambda, r}$ is regular.

As regards the application of $R_{\lambda, r}$ to (6), it is clear that for $\lambda_n = n$, $R_{\lambda, 1}$ gives a value at points inside and on the boundary of the circle of convergence, but nowhere outside. It can be proved that with any choice of (λ_n) and r, evaluation of (4) outside the circle of convergence is impossible.

The definitions thus far considered have all been of type G'. We consider next some which are of the more general type G. A definition given by Euler can readily be put in this form. Euler argued that as the series $1 - 1 + 1 - 1 + \cdots$ is the special case, for $t = 1$, of the series $1 - t + t^2 - t^3 + \cdots$, which for $t < 1$ has the value $1/(1 + t)$, therefore the former series should have the value $\frac{1}{2}$. This amounts to writing, in the case of a given series $u_1 + u_2 + u_3 + \cdots$,

$$y(t) = u_1 + u_2 t + u_3 t^2 + \cdots,$$

and defining the value as $\lim_{t \to 1} y(t)$. This definition may be called Euler's power series method. Thrown into a form involving x_n instead of u_n, it will read

$$(P) \qquad y_n = (1 - t) \sum_{k=1}^{\infty} x_k t^{k-1},$$

so that in the notation of G, $a_k(t) = (1 - t)t^{k-1}$. We must of course require that the series

$$\sum_{k=1}^{\infty} x_k t^{k-1}, \qquad \sum_{k=1}^{\infty} u_k t^{k-1}$$

converge for $|t| < 1$.

We may choose to let t approach 1 along real values or along some other point set T in the circle $|t| < 1$, having 1 as a limit point. In the first case, the fact that P is regular is stated by a well known theorem due to Abel: *If* $\Sigma \alpha_k t^{k-1}$ *converges*, $-1 < x \leqq 1$, *then it is continuous even at* $t = 1$.

Abel's theorem was extended by Stolz and Pringsheim to the case of other point sets T. The conditions shown by them to be sufficient are easily shown by the theorem of Carmichael and Hildebrandt to be also necessary: P is regular if T lies within the angle formed by some pair of chords through $t = 1$.

P evaluates (4) only at points other than $x = 1$, within and on the boundary of the circle of convergence. It is furthermore obvious that P can never evaluate any power series at a point outside its circle of convergence, and that it is therefore of little value for the problem of analytic extension.

Several elegant methods of evaluation were given by Borel. I shall state some of them in a more general form due to Sannia.* In terms of a given sequence (x_n), write

$$\xi(t) = \sum_{k=1}^{\infty} x_k \frac{t^{k-1}}{(k-1)!}$$

and suppose that this power series converges for all points of the plane. Call

$$\xi^{(0)}(t) = \xi(t),$$

$$\xi^{(1)}(t) = \frac{d}{dt}\xi^{(0)}(t), \qquad \xi^{(-1)}(t) = \int_0^t \xi^{(0)}(\tau)d\tau,$$

$$\xi^{(2)}(t) = \frac{d}{dt}\xi^{(1)}(t), \qquad \xi^{(-2)}(t) = \int_0^t \xi^{(-1)}(\tau)d\tau,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot.$$

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* RENDICONTI DI PALERMO, vol. 42 (1917), p. 303.

Define

$$(B_r) \qquad y(t) = e^{-t}\xi^{(r-1)}(t);$$

then $\lim_{t\to\infty} y(t)$ is to be the generalized limit of (x_n). The cases commonly discussed are B_1, known as the Borel mean definition, and B_0, known as the Borel integral definition.

In the usual form of treatment T is taken as the positive real axis, and it is shown that B_r is always regular. It seems not to have been noted that we may allow T to be a more general point set of the complex plane possessing points of arbitrarily great positive abscissas, and take the limit as $\mathbf{R}(t)$ becomes infinite. It can readily be proved that for the Borel-Sannia definition B_r to be regular it is necessary and sufficient that all points of T of sufficiently great positive abscissas lie within some parabola having its axis along the real axis and opening to the right; that is, if $t = \tau + i\sigma$, σ^2/τ shall be bounded.

Let us apply these results to (6). We have

$$\xi^{(0)}(t) = xe^{xt}, \qquad \xi^{(1)}(t) = x^2e^{xt}, \qquad \xi^{(2)}(t) = x^3e^{xt}, \qquad \cdots;$$

$$\xi^{(-1)}(t) = e^{xt} - 1, \qquad \xi^{(-2)}(t) = \frac{1}{x}e^{xt} - \frac{1}{x} - t, \qquad \cdots.$$

Hence

$$B_1\colon\ y(t) = xe^{(x-1)t}; \quad B_0\colon\ y(t) = e^{(x-1)t} - e^{-t};$$

$$B_2\colon\ y(t) = x^2e^{(x-1)t}; \quad B_{-1}\colon\ y(t) = \frac{1}{x}e^{(x-1)t} - \frac{1}{x}e^{-t} - te^{-t};$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ; \qquad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ .$$

Taking for T the positive real axis, we see that B_r evaluates (6) to zero for $\mathbf{R}(x) < 1$, and for no other values. By appropriate choice of other sets for T, we can secure the evaluation for any point of the line $\mathbf{R}(x) = 1$ except $x = 1$.

Borel also gave a definition which he termed "absolute summability." It is not regular; it does, however, correctly evaluate all absolutely convergent series. Absolute summability has been considered important since it permitted various operations on series which were not justified by mere summability B_1 or B_0; but with Sannia's form of the Borel definitions, absolute summability becomes relatively unimportant.

An extremely powerful definition is due to Le Roy. Write

$$y\ (t) = \sum_{k=1}^{\infty} \frac{\Gamma(\overline{k-1}\,t+1)}{(k-1)\,!}\,u_k,$$

taking for T the real axis near $t = 1$; and $t_0 = 1$. This definition can be shown to be regular. It evaluates (4) for all points of the complex plane except x real, $\geqq 1$.

The Riesz means have been extended by Hardy and Riesz so as to bring them under the form G. Taking the sequence (λ_n) restricted as before, write

$$(R_{\lambda,\,r}) \qquad y(t) = t^{-r} \sum_{\lambda_k < t} (t - \lambda_k)^r u_k,$$

and let T be the set of positive real numbers, $t_0 = \infty$. $R_{\lambda,\,r}$ in this form is still regular for all $r \geqq 0$. It does not evaluate (4) in any more extended region than did the form for $R_{\lambda,\,r}$ previously given.

4. *Relative Inclusiveness of Definitions; Equivalence.* A definition, A, is said to *include* another, B, in case every sequence summable B is summable A to the same value. Two definitions are said to be *equivalent* if each includes the other; in this case each evaluates exactly the same sequences.

If the definitions A, B, whether regular or not, are of type G', A includes B provided there exists a regular C such that $A = CB$. In case B possesses an inverse, this condition may be written in the form: AB^{-1} is regular; it is in this case necessary and sufficient.[*]

If A, B, whether regular or not, are of type G', they are equivalent provided there exist C, D, both regular, such that $A = CB$, $B = DA$. In case A, B possess inverses, this condition may be written in the form: AB^{-1} and BA^{-1} are regular; it is in this case necessary and sufficient.[†]

Certain cases fall under the criteria just given without any further investigation. For instance, since the Hölder means satisfy the condition $M^r M^s = M^{r+s}$, and since M^r is always regular, it is clear that the Hölder mean definition of any order

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[*] The second form of the condition is stated by Carmichael, loc. cit., p. 112; it has usually been applied in this form. There are cases, however, in which the first form is useful.

[†] Second form stated by Carmichael, loc. cit., p. 113.

includes that of any lower order. In a similar way the exponential mean E_r becomes more inclusive as r increases, since $E_rE_s = E_{rs}$ and E_r is regular when $r > 1$.

Other instances require more investigation, but every case I have examined in which definitions of type G' are concerned depends on the criterion I have stated. The Cesàro-Chapman mean C_r includes C_s if $\mathbf{R}(r) > \mathbf{R}(s) > -1$. In particular, for $-1 < \mathbf{R}(r) < 0$, C_r is included in I; thus every series summable C_r, $-1 < \mathbf{R}(r) < 0$, is convergent; and not all convergent series are summable C_r for a fixed r, $-1 < \mathbf{R}(r) < 0$.

A question which has formed the starting point for a number of investigations on divergent series is the relationship of M^r and C_r for positive integral values of r. Knopp proved that for any such r, C_r includes M^r; Schnee and Ford showed that M^r and C_r are actually equivalent; a number of other proofs have been given. To cite an example of the opposite kind, M^r and E_s are distinctly overlapping definitions; neither includes the other, except in the trivial case $M^0 = E_1 = I$.

It is important to point out that the method of proof of relative inclusiveness given above can be applied at times even to the comparison of definitions of different types. For instance, E_r is of type G', a sequence-to-sequence transformation, while B_1 is of type G, a sequence-to-function transformation. But it is easily proved that $B_1E_r^{-1} = B_1E_{1/r} = \mathbf{E}B_1$, where $\mathbf{E}$ is a function-to-function transformation:

$$(\mathbf{E}) \qquad y(t) = x(rt).$$

Thus B_1 will include E_r provided $\mathbf{E}B_1$ is regular; and this will surely be true provided $\mathbf{E}$ is regular, where the latter statement must be understood to mean that the existence of $\lim_{t\to\infty} x(t)$ is to imply the existence and equality of $\lim_{t\to\infty} y(t)$. But the regularity of $\mathbf{E}$ in the sense explained is obvious; thus B_1 includes E_r for every $r > 0$.

Into similar form may be thrown Sannia's proof that B_r includes B_s whenever $r < s$, and the familiar proof that $C_r (r > 0)$ is included in P. On the other hand, of M^r and B_1 neither includes the other; the same statement holds for B_1

and P. Note that no definition which evaluates a power series at a point outside its circle of convergence can include P.

The results for the Riesz means are interesting. As in the case of the Cesàro means, $R_{\lambda,\,r}$ includes $R_{\lambda,\,s}$ if $r > s$. We may even compare means derived from different sequences (λ_n); thus, if $\mu_n = \log \lambda_n$, then $R_{\mu,\,r}$ includes $R_{\lambda,\,r}$. If $\lambda_n = n$, $R_{\lambda,\,r}$ is equivalent to C_r for every r. In case $\lambda_n = e^n$, $R_{\lambda,\,r}$ is always equivalent to I.

5. *Mutual Consistency.* It has been seen that two definitions may be regular and yet such that neither includes the other. In such cases it is manifestly of the highest importance to know that the two definitions will not give different values to any sequence which each one evaluates. The circumstance in question is illustrated by an example due to Silverman.* The definitions (of type G'):

$$y_n = \frac{1}{n}\sum_{k=1}^{n} x_k, \qquad y_n = \frac{1}{n}\sum_{k=2}^{n}\left[1 + \frac{(-1)^{k+1}}{\log k}\right] x_k$$

are both regular; but the sequence $x_n = (-1)^n \log n$ is evaluated by the first to 0 and by the second to 1.

We call two definitions *mutually consistent*† if, whenever each of them evaluates a sequence, the two values are the same. A condition for mutual consistency can be stated as follows:

Any two definitions are mutually consistent if there exists a definition which includes each of them.

Obvious as this criterion seems, it is nevertheless of real value. An important special case is: *Two definitions A, B of type G' are mutually consistent if there exist C, D, both regular, such that $CA = DB$.* In fact, the definition expressed by either of the two equal forms CA, DB includes both A and B.

Still further specialization occurs in case it happens that we can choose $C = B$, $D = A$. *Any two regular definitions of type G' are mutually consistent if they are permutable.*

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* Loc. cit., p. 38.

† In a paper by Silverman and myself (TRANSACTIONS OF THIS SOCIETY, vol. 18 (1917), p. 1), the word "consistent" was used for this idea. The term in the text, suggested by Carmichael, loc. cit., p. 111, seems preferable.

It can be shown* that all transformations of type G' permutable with M are permutable with each other; hence *all definitions of type G' permutable with M are mutually consistent.*

An interesting case of this kind is furnished by E_r. We have seen that E_r and M are overlapping definitions; but it is easily verified that E_r is permutable with M, hence E_r and M^s are mutually consistent.

The method outlined above, like that in the preceding section, can sometimes be applied even if the two definitions are of different types. Thus, it can be shown that $B_1M = \mathbf{M}B_1$, where **M** is the function-to-function transformation

$$(\mathbf{M}) \qquad y(t) = \frac{1}{t}\int_0^t x(u)du,$$

and where all the formal processes involved have a meaning and are correct if applied to a sequence summable both B_1 and M. Thus B_1 and M will be consistent if **M** is regular; that is, if the existence of $\lim_{t\to\infty} x(t)$ implies the existence and equality of $\lim_{t\to\infty} y(t)$. The transformation **M** is exactly what is known as the first Hölder (or Cesàro) mean for continuous limits;† it is known to be regular. Equally simple is the proof that M^r and B_1 are mutually consistent.

This method of proof seems susceptible of wide application in studying the question of mutual consistency, which is important, and which has as yet received little attention.

6. *Total Regularity.* Let us now suppose we are dealing only with real sequences, and applying to them only real transformations. A regular definition must evaluate any convergent sequence, giving it its true value. It may naturally be asked whether this conservation of finite limits applies also to infinity of definite sign; in other words, whether a sequence becoming (say) positively infinite need be evaluated by a regular definition to positive infinity. The definition $2M - I$:

$$y_n = 2\frac{x_1 + x_2 + \cdots + x_n}{n} - x_n$$

is obviously regular; but if we take $x_n = n$, then we find

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* Hurwitz and Silverman, loc. cit., p. 7.

† See § 10.

$y_n = 1$, so that

$$\lim_{n\to\infty} x_n = +\infty, \qquad \lim_{n\to\infty} y_n = 1.$$

In this case therefore regularity does not extend to the conservation of the improper limit $+\infty$.

A regular definition may be called *totally regular* if it evaluates every sequence which becomes positively (negatively) infinite to $+\infty$ $(-\infty)$.

A sufficient condition that a regular definition G be totally regular is that for all sufficiently great values of k, $a_k(t) \geqq 0$. In this case, condition (G2) of the conditions for regularity is superfluous. If the definition is of type G', the condition, $a_{nk} \geqq 0$ for all sufficiently great values of k, is also necessary.

A closely related consideration is that of the effect of a regular transformation on the limits of indeterminacy of a sequence which it does not evaluate. It is desirable that a definition, if it does not evaluate a specific sequence, shall at least not render its oscillation more violent; this may readily happen, however. If we apply the definition $2M - I$ to the sequence $0, 2, 0, 4, 0, 8, \cdots$, for which

$$\liminf_{n\to\infty} x_n = 0,$$

we find that

$$\liminf_{n\to\infty} y_n = -\infty.$$

Without endeavoring to answer completely the question raised, I shall merely say that the condition for total regularity is sufficient also to insure that the new limits of indeterminacy shall not fall outside the interval of the old limits.

The criterion for total regularity is easily tested for all the usual definitions. It is found that M^r, C_r, $R_{\lambda,r}$ are totally regular for the values of $r \geqq 0$ for which they are defined; E_r is totally regular for $r \geqq 1$. When the point set T is real, P, B_r and the definition of Le Roy are totally regular.*

We may also widen the scope of the notions of relative inclusiveness and equivalence to take account of sequences evaluated to $+\infty$. Obviously, on account of the relations $M^rM^s = M^{r+s}$, $E_rE_s = E_{rs}$, the statements previously made regarding relative inclusiveness of M^r for varying r and of E_r for varying r remain true even in the present extended sense. But it can be shown that M^r and C_r, which are equivalent for finite limits when r is a positive integer, do not retain this equivalence for the limit $+\infty$ when $r \geqq 2$; in fact there will always be sequences which are evaluated by M^r to $+\infty$, and are not so evaluated by C_r.

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* The consideration of total regularity settles the question of the effect of these definitions on the series $1 + x + x^2 + \cdots$ at the point $x = 1$, which was left open in § 3.

Silverman showed that the criterion for total regularity has the following consequences: No definition of type G' possessing an inverse can be equivalent to I both as regards finite and definitely infinite limits unless it is of the form K: $y_n = c_nx_n$, where $\lim_{n\to\infty} c_n = 1$. No two definitions A, B of type G' possessing inverses can be equivalent both as regards finite and definitely infinite limits unless $A = KB$.

These statements would not hold if the restriction as to the possession of inverses were removed, as may be seen from the trivial example $y_1 = 0$; $y_n = x_{n-1}, .n > 1$; which is equivalent to I, even in the present extended sense.

7. *Adjunction or Omission of Elements.* If a sequence converges, then the new sequence obtained by prefixing or omitting an element at the beginning will converge to the same value. We may inquire whether a similar property holds for definitions of summability; if a sequence is summable A to a value l, will the sequence obtained by prefixing or omitting an element be summable A (or summable B, where B is expressible by means of A) to the value l?

This question has been answered for a number of definitions. In so far as it relates to prefixing an element, the answer must be independent of the value of the element prefixed; only the alteration in rank is significant. The question is equivalent to that of prefixing or omitting a term at the beginning of a series, with appropriate alteration in value of the series; it is in this form that it has usually been studied.

I shall summarize the most important results. M^r, C_r, E_r, whenever they are regular, and P, permit adjunction or omission of an element. Borel's "absolute summability," which is not regular, has the same property; a more satisfactory settlement of the question for the Borel means is given by the theorem of Sannia: If a sequence is summable B_r, then the sequence obtained by omitting an element is summable to the same value at any rate by the stronger definition B_{r-1}, and the sequence obtained by omitting an element is summable to the same value even by the weaker definition B_{r+1}.

Hardy and Riesz state that it is possible to have a series summable $R_{\lambda,\, r}$ and remain so summable when a term is omitted, the two values not differing by an amount equal to the omitted term.

8. *Necessary Conditions for Summability.* If a series Σu_n converges, then u_n must approach zero; this is of course not sufficient for convergence, but it is a very useful property of convergent series. Somewhat similar necessary conditions exist for summability with respect to the commoner definitions; these conditions are sometimes expressed so as to involve terms of the series, sometimes elements of the sequence. For summability M^r or C_r, a necessary condition is that $\lim u_n/n^r = 0$, and in fact even that $\lim x_n/n^r = 0$; for summability E_r it is necessary that $\lim u_n/(2r-1)^n = 0$ and $\lim x_n/(2r-1)^n = 0$.

For summability P it is necessary that the series $\Sigma x_n t^{n-1}$ and $\Sigma u_n t^{n-1}$ have radii of convergence $\geqq 1$, therefore that $\limsup |x_n|^{1/n} \leqq 1$ and $\limsup |u_n|^{1/n} \leqq 1$. For summability B_r it is necessary that the series $\Sigma x_n t^{n-1}/(n-1)!$ have infinite radius of convergence, hence that $\lim |x_n|^{1/n}/n = 0$ and $\lim |u_n|^{1/n}/n = 0$. A necessary condition that a sequence (x_n) be summable $R_{\lambda,\, r}$ to l is that

$$\lim_{n\to\infty} \left(\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right)^r (x_n - l) = 0.$$

9. *Multiplication of Series.* The totality of expressions obtained by multiplying terms of one series Σu_n by terms of another series Σv_n may be represented formally by a double series $\Sigma u_m v_n$. If we collect the terms of this double

series in any way into a simple series Σw_n, the latter may be called a product series of the two given series. The behavior of the new series if the original series converges will differ according to the way in which the terms are grouped. Thus, if

$$\begin{aligned} w_1 &= u_1v_1, \\ w_2 &= u_1v_2 + u_2v_2 + u_2v_1, \\ w_3 &= u_1v_3 + u_2v_3 + u_3v_3 + u_3v_2 + u_3v_1, \\ &\cdots\cdots\cdots\cdots, \end{aligned}$$

Σw_n will converge whenever Σu_n, Σv_n converge.

Other methods of grouping, however, are generally more useful. The so-called Cauchy product, suggested by the grouping most natural for power series, takes

$$\begin{aligned} w_1 &= u_1v_1, \\ w_2 &= u_1v_2 + u_2v_1, \\ w_3 &= u_1v_3 + u_2v_2 + u_3v_1, \\ &\cdots\cdots\cdots\cdots. \end{aligned}$$

In this case convergence of Σu_n, Σv_n does not insure convergence of Σw_n; it is, however, true that if Σu_n, Σv_n are convergent, then Σw_n will be summable M to the product of the values of Σu_n, Σv_n. More generally, the Cauchy product of two series summable respectively C_r and C_s ($\mathbf{R}(r) > -1$, $\mathbf{R}(s) > -1$) is summable C_{r+s+1}, and the value of the product series is the product of the values of the given series.

This gives an interesting instance of the use of non-regular definitions. If two series are not merely convergent, but summable $C_{-\frac{1}{2}}$, then their Cauchy product will be convergent.

There is a similar theorem for the Borel-Sannia definition. If two series are summable B_r and B_s, their Cauchy product is summable B_t to the correct value, where

$$t = \begin{cases} r + s - 1 \text{ unless } r > 0, \text{ or } s > 0; \\ \text{the lesser of } r, s \text{ if } r > 0, \text{ or } s > 0. \end{cases}$$

The multiplication of Dirichlet's series suggests a different grouping of terms:

$$w_n = \sum_d u_d v_{n/d},$$

where d takes as its values the divisors of n. For this grouping, which we may call the Riesz product, we have the result: The Riesz product of two series summable $R_{\lambda,r}$ and $R_{\lambda,s}$, where $\lambda_n = \log n$, $r > 0$, $s > 0$, is summable $R_{\lambda,r+s+1}$.

10. *Extensions to Other Types of Limit.* The endeavor to assign a meaning to $\lim_{n\to\infty} x_n$ when it does not exist in the ordinary sense leads naturally to the same attempt for $\lim_{t\to\infty} x(t)$, where $x(t)$ is a function of the continuous real variable t. The analogues of the Cesàro and Hölder means were studied by Landau.* Investigations of the general type corresponding to G' were made by Silverman† and Kojima.‡ I shall not repeat the most general results of these authors, but shall quote only the following special case:§

If $K(x, y)$ *is integrable in* y *for each value of* x, $0 < y \leqq x$, *and if for any function* $u(x)$ *which is bounded and integrable in any finite interval*, $x \leqq 0$, *we define*

$$v(x) = \alpha u(x) + \int_0^x K(x, s)u(s)ds,$$

then a sufficient condition that $\lim_{x\to\infty} v(x) = l$ *whenever* $\lim_{x\to\infty} u(x) = l$ *is that for constant* a,

$$\int_0^a |K(x, y)|dy \text{ converges}, \qquad \lim_{x\to\infty} \int_0^a |K(x, y)|dy = 0,$$

that for $x > 0$,

$$\int_0^x |K(x, y)|dy$$

is bounded, and that

$$\lim_{x\to\infty} \int_0^x K(x, y)dy = 1 - \alpha.$$

Extensions to double series have also been made. The analogues of the Cesàro and Hölder means were given by C. N. Moore.‖ Robison¶ has made a general study of regularity. An element of novelty in comparison to the case of simple series is that even a convergent double series need not have its terms bounded. Robison's result corresponding to the theorem of Silverman and Toeplitz for simple series follows.

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* SÄCHSISCHE BERICHTE, vol. 65 (1913), p. 131.

† TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 284; this BULLETIN, vol. 22 (1915–16), p. 459.

‡ TÔHOKU JOURNAL, vol. 14 (1918), p. 64; vol. 18 (1920), p. 37.

§ Silverman, this BULLETIN, loc. cit.

‖ TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), p. 73.

¶ Ph.D. thesis, Cornell, 1919.

A necessary and sufficient condition that the transformation

$$y_{mn} = \sum_{k=1,\, l=1}^{m,\, n} a_{mnkl}\, x_{kl}$$

carry every bounded convergent double sequence (x_{mn}) *into a bounded double sequence* (y_{mn}) *convergent to the same value is that*

(1) *for each k, l,* $\lim_{m\to\infty,\, n\to\infty} a_{mnkl} = 0;$

(2) *for all m, n,* $\sum_{k=1,\, l=1}^{m,n} |a_{mnkl}|$ *is bounded;*

(3) *for each l,* $\lim_{m\to\infty,\, n\to\infty} \sum_{k=1}^{m} |a_{mnkl}| = 0,$

and for each k, $\lim_{m\to\infty,\, n\to\infty} \sum_{l=1}^{n} |a_{mnkl}| = 0;$

(4) $\lim_{m\to\infty,\, n\to\infty} \sum_{k=1,\, l=1}^{m,\, n} a_{mnkl} = 1.$

Robison has given also the theorem for double series corresponding to the result of Carmichael and Hildebrandt, and the condition for total regularity.

11. *Other Questions.* Time does not permit a detailed account of other interesting lines of study in connection with divergent series; a brief mention of a few results must suffice.

Closely related to the regularity of a transformation of type G is the requirement that it carry every convergent sequence into a convergent sequence (irrespective of any relationship between the two limits); or that it carry a bounded sequence into a convergent sequence, or a bounded sequence into a bounded sequence. These conditions have been studied by Kojima,* Fraleigh,† and J. Schur,‡ and for double sequences by Robison.|| Interesting investigations have been made of properties possessed not by all series summable according to a certain definition, but only by such of them as satisfy further conditions. Typical theorems of this kind are the following:

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* Tôhoku Journal, vol. 12 (1917), p. 291.
† A.M. thesis, Cornell, 1918.
‡ Journal für Mathematik, vol. 151 (1920), p. 79.
|| Loc. cit.

If Σu_n *is summable M and* $nu_n < K$, *then* Σu_n *is convergent.*[*]

If Σu_n *is summable* B_1 *and* $\sqrt{n}\,|u_n| < K$, *then* Σu_n *is convergent.*[†]

For applications of the theory of divergent series to important special types of series, to differential equations, and to mathematical physics, reference may be made to the three expository papers mentioned in § 1.

12. *Conclusion.* I shall permit myself, in closing, to make two observations which represent only personal opinion.

Any definition of the generalized limit of a sequence is ultimately only an actual limit of something else; it seems to me worth while to recall frequently, in dealing with divergent series, that we are in fact studying only ordinary processes of convergence. It is at times more illuminating for the comprehension of a theorem on summable series to supply all the transformations implied in the definition of summability and state the result entirely in terms of ordinary limits than to use the more concise form which is in essence symbolic. Indeed, important applications of the conditions for regularity are proofs of theorems on limits, in which divergent sequences present themselves, if at all, only as an afterthought. Such applications have been given by Silverman and by Schur.

As regards the various current problems in connection with divergent series, the most important seems to me personally to be that of mutual consistency. It would be desirable to be able to assert of any two definitions which have been used practically that they are or are not mutually consistent, and to have such workable criteria as would make it possible to test new definitions which may be proposed. Without such information, the use of two different methods of summability in a single investigation, unless one is merely included in the other, would seem to produce at least grave inconvenience.

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[*] Hardy, PROCEEDINGS OF THE LONDON SOCIETY (2), vol. 8 (1910), p. 302; Landau, PRACE MATEMATYCZNO-FIZYCZNE, vol. 21 (1910), p. 97; Fujiwara, TÔHOKU JOURNAL, vol. 15 (1919), p. 323.

[†] Hardy and Littlewood, RENDICONTI DI PALERMO, vol. 41 (1916), p. 36.

NOTE ON AN IRREGULAR EXPANSION PROBLEM*

BY DUNHAM JACKSON

One of the simplest expansion problems depending on a set of boundary conditions of the type which Birkhoff has called *irregular*, is that associated with the system

$$\frac{d^3u}{dx^3} + \rho^3 u = 0, \tag{1}$$

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(\pi) = 0.$$

Let $u_1(x)$, $u_2(x)$, $\cdots$ be the characteristic functions of this differential system. The writer showed, a number of years ago,† that the formal expansion of an arbitrary continuous function $f(x)$ in a series of the form

$$a_1u_1(x) + a_2u_2(x) + \cdots \tag{2}$$

will ordinarily be divergent, even if $f(x)$ satisfies conditions which would insure the convergence of its development in terms of the characteristic functions of a regular system; and it may diverge even if $f(x)$ is analytic throughout the interval $(0, \pi)$. A subsequent paper by Hopkins‡ specified a restricted class of analytic functions to which $f(x)$ must belong, if the expansion is to converge uniformly to the desired value. He showed that the necessary condition thus obtained, if supplemented by certain secondary hypotheses, is also sufficient. He did not discuss the question whether the formal expansion, if uniformly convergent at all, must necessarily have $f(x)$ for its sum. His analysis, however, makes it possible to recognize without difficulty that this is the case. It is the purpose of the following paragraphs to supply the details of the demonstration.*

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* Presented to the Society, April 23, 1921.

† D. Jackson, *Expansion problems with irregular boundary conditions*, Proceedings of the American Academy, vol. 51 (1915–16), pp. 383–417; see pp. 384–393.

‡ J. W. Hopkins, *Some convergent developments associated with irregular boundary conditions*, Transactions of this Society, vol. 20 (1919), pp. 245–259.

The system adjoint to (1) is

$$\frac{d^3v}{dx^3} - \rho^3 v = 0, \tag{3}$$

$$v(0) = 0, \qquad v(\pi) = 0, \qquad v'(\pi) = 0.$$

Its characteristic functions may be denoted by $v_1(x)$, $v_2(x)$, $\cdots$, and then the coefficients in the series (2) will be given by

$$a_n = \frac{1}{q_n}\int_0^\pi f(x)v_n(x)dx, \qquad q_n = \int_0^\pi u_n(x)v_n(x)dx.$$

If the series converges uniformly, its sum $g(x)$ will be a continuous function vanishing for $x = 0$ and for $x = \pi$ (and satisfying also the condition $g'(0) = 0$, though it is not necessary to insist on that point here). By the orthogonal properties of the system of functions $u_n(x)$, $v_n(x)$, we see that

$$\frac{1}{q_n}\int_0^\pi g(x)v_n(x)dx = a_n,$$

so that the difference

$$r(x) = f(x) - g(x)$$

has the property that

$$\int_0^\pi r(x)v_n(x)dx = 0 \qquad (n = 1, 2, \cdots). \tag{4}$$

We shall assume at first that $f(x)$ vanishes at both ends of the interval $(0, \pi)$, and then $r(x)$ will have this property also. It is to be shown that $r(x)$ must vanish identically.

It follows from Hopkins's work† that the formal development of

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* This proof is more or less in line with a conversation between Professor Birkhoff and the writer, which took place early in 1917, before Mr. Hopkins's work was completed. Neither party to the conversation can remember definitely now what was said, except that it probably covered a part, but certainly not the whole, of what is done here. The fundamental idea of the proof has been used frequently in other connections.

† Loc.cit., Theorem III, and p. 259, footnote. The proof can also be based directly on § 3 of the paper, without reference to the more general method of § 4.

$$x^{3k+2} - \pi^{3k}x^2$$

in a series of the form (2) converges uniformly for $0 \leqq x \leqq \pi$, if k is any positive integer. Since the system (1) and the system (3) are carried over into each other by the substitution of $\pi - x$ for x, and, in particular*

$$v_n(x) = u_n(\pi - x),$$

it follows that

$$p_k(x) = (\pi - x)^{3k+2} - \pi^{3k}(\pi - x)^2 \tag{5}$$

can be expanded in a uniformly convergent series of the functions $v_n(x)$.

It will be shown that $r(x)$ can be approximately represented, uniformly for $0 \leqq x \leqq \pi$, with any desired degree of accuracy, by means of a linear combination of a finite number of the expressions $p_k(x)$. Hence $r(x)$ can be similarly represented in terms of the functions $v_n(x)$, each $p_k(x)$ being replaced by a sufficient number of terms of its expansion in series of the v's.

The hypothesis has already provided that $r(0) = r(\pi) = 0$. If $r_1(x)$ is defined as equal to $r(x)$ for $0 \leqq x \leqq \pi - 2\delta$, equal to $(\pi - x)^2$ for $\pi - \delta \leqq x \leqq \pi$, and linear and continuous for $\pi - 2\delta \leqq x \leqq \pi - \delta$, then $r_1(x)$ will differ arbitrarily little from $r(x)$ throughout $(0, \pi)$, if δ is taken sufficiently small, and will have the form

$$r_1(x) = (\pi - x)^2 q(x), \tag{6}$$

where $q(x)$ is continuous for $0 \leqq x \leqq \pi$, and vanishes for $x = 0$. The quantities involved being real, $\pi - x$ is a single-valued continuous function of $(\pi - x)^3$, and the same is true of x. Hence $q(x)$ is a single-valued continuous function of $(\pi - x)^3$ for $0 \leqq x \leqq \pi$. Therefore by Weierstrass's theorem, it can be approximately represented with any desired accuracy by a polynomial in $(\pi - x)^3$. That is, if ϵ is any positive quantity, there will exist an exponent m and a set of coefficients

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* Cf., e.g., D. Jackson, loc. cit., pp. 387, 389.

$A_0, A_1, \cdots, A_m$, so that

$$|A_0 + A_1(\pi - x)^3 + A_2(\pi - x)^6 + \cdots + A_m(\pi - x)^{3m} - q(x)| < \epsilon$$

throughout $(0, \pi)$. In particular, for $x = 0$, since $q(0) = 0$,

$$|A_0 + A_1\pi^3 + A_2\pi^6 + \cdots + A_m\pi^{3m}| < \epsilon.$$

Hence, by subtraction within the bars,

$$|A_1[(\pi - x)^3 - \pi^3] + \cdots + A_m[(\pi - x)^{3m} - \pi^{3m}] - q(x)| < 2\epsilon;$$

from (5) and (6),

$$|A_1p_1(x) + \cdots + A_mp_m(x) - r_1(x)| < 2\pi^2\epsilon;$$

and the existence of the desired approximation for $r(x)$ can be inferred at once.

It is to be regarded as established, then, that $r(x)$ can be represented with any required accuracy by a linear combination of the v's, or, what amounts to the same thing, that $r(x)$ can be expanded in a uniformly convergent series of the form

$$r(x) = C_1V_1(x) + C_2V_2(x) + \cdots,$$

where $V_n(x)$ is a linear combination of $v_1(x), v_2(x), \cdots, v_n(x)$, with constant coefficients. Because of (4),

$$\int_0^\pi r(x)V_n(x)dx = 0 \qquad (n = 1, 2, \cdots).$$

Consequently

$$\int_0^\pi [r(x)]^2dx = \int_0^\pi r(x)[C_1V_1(x) + C_2V_2(x) + \cdots]dx = 0,$$

and $r(x)$, being continuous, must vanish identically.

It has been assumed hitherto that $f(x)$ vanishes at the ends of the interval. This must in fact be the case, if $f(x)$, always supposed continuous, is to give rise to a uniformly convergent expansion. Let us make the contrary hypothesis, and show that it leads to a contradiction. The continuous function $g(x)$ must still vanish at 0 and π, and $r(x)$ will have the same value as $f(x)$ at these points, being accordingly different from zero at one or both of them. It will then be no longer possible to approach $r(x)$ uniformly by means of linear combinations of the v's. If ϵ is any positive quantity, however, it will be possible to choose a function $r_2(x)$, continuous throughout the interval and vanishing at 0 and π, in such a way that

$$\int_0^\pi |r(x) - r_2(x)|\,dx < \frac{\epsilon}{2}\cdot$$

Then $r_2(x)$, by the reasoning applied to $r(x)$ under the previous hypothesis, can be approached by a linear combination of the v's, from v_1 to v_n, say, with a maximum error not exceeding $\epsilon/(2\pi)$. If the approximating function is denoted by $W_n(x)$ we shall have

$$\int_0^\pi |r(x) - W_n(x)|\,dx < \epsilon.$$

If we do this for a succession of values of ϵ approaching zero, we obtain a succession of functions* $W_n(x)$, such that

$$\lim_{n\to\infty}\int_0^\pi |r(x) - W_n(x)|\,dx = 0.$$

Let M be the maximum of $|r(x)|$; then

$$\left|\int_0^\pi r(x)[r(x) - W_n(x)]dx\right| \leqq M\int_0^\pi |r(x) - W_n(x)|\,dx,$$

so that

$$\lim_{n\to\infty}\int_0^\pi r(x)[r(x) - W_n(x)]dx = 0.$$

Since $\int rW_n dx = 0$ for all values of n, it is seen again that $\int r^2 dx = 0$, and $r(x)$ vanishes identically. This is at the same time a verification of the principal result to be established, and a proof that the hypothesis with regard to the non-vanishing of $f(x)$ at the ends of the interval was inadmissible.

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* It is allowable to think of the index n as taking on all positive integral values, since a linear combination of the first n of the v's is at the same time a linear combination of the first n' of them, if n' is any number greater than n.

THE BÔCHER MEMORIAL PRIZE

The establishment of the Bôcher Memorial Prize is a noteworthy departure in American Mathematical polity. It is believed that this is the first mathematical prize in our country to be given at regular intervals for research in pure mathematics.

The inception of the prize was due to Professor T. S. Fiske who, after the death of Maxime Bôcher in 1918, raised a voluntary fund for a permanent memorial of his work and services. This fund was turned over to the American Mathematical Society, and a special committee consisting of T. S. Fiske, C. N. Haskins, Dunham Jackson, O. D. Kellogg, and E. B. Van Vleck was appointed to consider its disposition. This committee reported to the Council the following resolutions which were adopted:

1. A prize of $100 (later a larger sum) shall be awarded once in every five years for a notable research-memoir, published in the Transactions of the American Mathematical Society during the preceding five years by a resident of the United States or Canada. The prize shall be known as the Bôcher Memorial Prize.

2. Such interest of the Memorial Fund as is not used for the prize shall be added to the principal.

3. The age of the recipient of the prize shall not be over forty years, and the prize shall not be awarded twice to the same person.

4. The first award of $100 shall be made by a Committee of the Council for a memoir published during the period 1918–1922, and shall be conferred at the annual or other designated meeting of the Society in 1923.

The suitability of the memorial will be apparent to all who are conversant with the history of American mathematics. Before many of the older generation imagined any immediate possibility of developing American mathematics to a permanent position of international importance, a group of the younger men had actually achieved this. It may be doubted whether another such formative period in American mathematics will ever recur. Professor Bôcher took his doctor's degree at the University of Goettingen in 1891, and on his return to Harvard University, became at once one of the outstanding figures in the new movement, thus stamping his impress on American mathematics when in its molten stage. When he died at the age of fifty-one, he left behind him a larger and more important aggregate of investigation in pure mathematics than had any preceding mathematician in America.

As the mathematical renaissance progressed, it became evident that the means for the publication of research in this country were altogether inadequate, and a new journal, the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, was founded through the persistence of a small group of men, among whom Professor Bôcher was conspicuous. Because of his unusual fitness for editorial work he was chosen as its second editor-in-chief and served for two successive terms, 1907–1913. His colleagues on the editorial board can testify to Professor Bôcher's extraordinary capacity for editorial work,—his quick and accurate judgment of memoirs, his unselfish and lavish expenditure of time, his clearness and elegance of presentation.

In determining what disposition should be made of the fund, many possibilities were discussed, and the final form of the recommendation was due to a combination of considerations. Because of the eminence of the scientific work of Professor Bôcher and his power as teacher to stimulate investigation, it was felt that there could be no more fitting memorial than the use of the fund to encourage research. Also, in recognition of his editorial ideals and service, it was desired to connect the memorial with the Transactions and thus assist in raising the quality of mathematical production in America and Canada.

Unfortunately, the fund is not large, $1200. In consequence, it was thought that a considerable portion of the interest must be set aside to secure the permanent usefulness of the fund. Hence it was recommended that the award of the prize should be made only once in five years, and that initially its amount should be limited to $100. For the present the unique honor of its receipt must compensate for its inadequacy. Would that some friend of Professor Bôcher and American mathematics would increase the size of the prize and thereby its importance!

Under conditions prevailing in our colleges and universities an extraordinary amount of elementary instruction is necessary. This routine work is so important, so time-consuming and energy-exhausting as to distract the attention of our brightest young instructors from research. The first ten years following the doctor's degree are indeed very critical ones, and for this reson an age limit has been imposed on the prize. In recognizing particularly the younger men, the desire is to help keep American mathematics ever young. The age limit of forty years may be too high, but continued production is as essential as early production. It is hoped that the memoirs honored by the prize not only will have much intrinsic value but will exhibit also some of Professor Bôcher's lucidity and directness of style.

While general conditions have been framed for the prize, the administration and interpretation of the above resolutions are left to the successive committees of award.

MADISON, WISCONSIN,
February 28, 1922.

EDWARD B. VAN VLECK.

ON KAKEYA'S MINIMUM AREA PROBLEM*

BY W. B. FORD

1. *Introduction.* During recent years the Japanese school of mathematicians, notably Professors Hayashi, Kakeya and Fujiwara, have proposed and investigated to some extent a unique and apparently new class of maxima-minima problems of which the one considered in this paper may be regarded as the simplest type. In general, such problems concern the determination of the closed curve of least area within which a given configuration may be completely rotated. The special problem in which we shall be interested appears to have been first stated by Kakeya and is as follows:†

A line-segment AB lying in the plane MN is to be moved so that it shall return to its original position but with its ends reversed (as in the rotation of a segment about its middle point through a semicircumference). How should this be done in order that the area generated during the motion may be a minimum?

2. *Interpretations of the Problem.* As thus stated, we note first that the problem admits of the following two interpretations: In computing area generated during any portion of the motion, the area S bounded by any given enclosure in the plane MN is to be counted (a) as many times as it is passed over by AB; (b) never more than once.

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* Presented to the Society, September 7, 1920.

† The existing literature upon this and the more general problems above referred to appears to be chiefly confined to the following three papers: *On the curves of constant breadth, and the convex closed curves inscribable and revolvable in a regular polygon,* by Tsuruichi Hayashi, Tôhoku Science Reports, vol. 5, pp. 303–312 (Dec. 1916); *On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle,* by M. Fujiwara and S. Kakeya, Tôhoku Journal, vol. 11, pp. 92–110 (Feb. 1917); *Some problems on maxima and minima regarding ovals,* by Soichi Kakeya, Tôhoku Science Reports, vol. 6, pp. 71–88 (July, 1917). Recently Pál (Mathematische Annalen, vol. 83 (1921), pp. 311–319) has given a complete solution, but under greater restrictions than those of this paper.

These two interpretations are illustrated in Fig. 1 wherein AB has been given a simple rotation of angle θ about a fixed point O lying in the perpendicular bisector CD of AB. Here A and B describe arcs of one and the same circle, thus taking the final positions A', B', while the segment AB passes over the singly shaded areas R, T once each, but passes over the doubly shaded area S twice, first by the portion CB and later by CA. Hence, for such a motion the area generated according to interpretation (a) is $R + 2S + T$, while in interpretation (b) it is $R + S + T$.

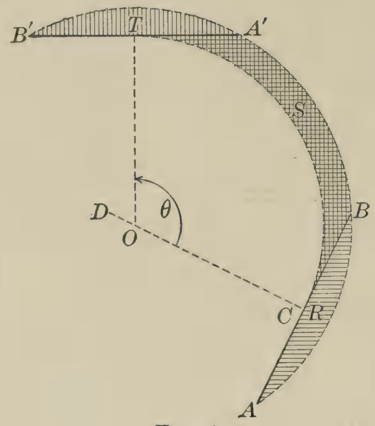

FIG. 1.

In order to make the necessary distinction thus arising, we shall hereafter refer to area generated in the sense (a) as area swept *over*, and to that generated in the sense (b) as area swept *out*.* We proceed, therefore, to consider the problem under interpretation (a) and it is believed that the method followed leads to a complete solution in this case.

3. *Infinitesimal Rotation.* Instead of undertaking directly the problem of § 1 wherein AB is to be turned completely end for end and finally brought back upon itself, we shall find it desirable to begin by considering the following more general yet in some respects more simple question.

How should a line-segment of length $2l$ be moved in such a way that the angle between its initial and final directions shall be a given amount, θ, while the area swept *over* (§ 2) shall be a minimum? Thus, suppose that AB is the initial position, its direction being regarded as from A to B, and let

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* In the studies of Kakeya and others already referred to, the term "area generated" is taken in the sense (b) only, this being the case of greatest complexity and interest, but inasmuch as our method for the study of (b) depends essentially upon that for the more simple case (a), we shall find it desirable to develop the latter first.

CD be a line whose direction (from C to D) makes the given angle θ with AB. The question then is, how should AB be given the same direction as CD in such a way as to sweep over a minimum area?* This question may be answered directly by use of the following simple kinematical principle which, for brevity, we shall assume without proof.

If a line-segment AB of length $2l$ lying in a plane MN is given an infinitesimal rotation of angle $d\theta$ about a fixed point O in MN, the area swept over will be less when O lies upon the perpendicular bisector of AB than when it lies at any point elsewhere.†

It thus appears, as regards the question proposed above, that for the desired minimum it is necessary and sufficient that during the motion each infinitesimal rotation shall be about some point in the perpendicular bisector of the segment. In fact, the infinitesimal area then swept over by an increment $d\theta$ in direction will be less than that obtained upon any other plan yielding the same change in direction, hence the same will be true of the sum of such infinitesimal areas and likewise of the limit of this sum, which limit is the area in question. In the customary language of kinematics, this means that the instantaneous center of motion should lie at all times upon the perpendicular bisector.

Moreover, since each infinitesimal area corresponding to a change in direction thus comes to differ from the value $l^2 d\theta$ by an infinitesimal of higher order than the first as compared to $d\theta$ as readily appears, it follows by Duhamel's theorem that the minimum area itself will have the value

$$l^2 \int_0^\theta d\theta = l^2\theta.$$

---

* We assume throughout that during the motion the angle which the segment makes with its initial direction increases only monotonically, as otherwise negative areas would be generated. However, θ is not restricted to the range $0 < \theta \leqq 2\pi$, but may be assigned any positive value whatever.

† The proof is readily carried out. In case the rotation takes place about a point in the perpendicular bisector the area swept over differs by an infinitesimal of higher order than $d\theta$ from $l^2 d\theta$, while if the rotation takes place about a point whose distance is h ($h > 0$) from the perpendicular bisector, the area swept over differs by an infinitesimal of higher order than $d\theta$ from $(l^2 + h^2)d\theta$.

Evidently there are an infinite number of ways of the type just described for moving AB into parallelism with CD. Of these the simplest is that in which AB is given a pure rotation of angle θ about its middle point regarded as fixed. The next simplest case is that in which AB is given a pure rotation of angle θ about some point in its perpendicular bisector other than its own middle point, thus sweeping over a circular strip such as shown in Fig. 1. However, it is to be observed that in general a movement such as we are considering will consist of both a rotation and a translation. In this connection the following general statement is noteworthy: Let AB be the initial position, C being the middle point. Draw any curve to which AB is tangent at C and such that, as one passes along the curve from C, the angle between its tangent and the initial direction never diminishes. Then, in order that AB shall sweep over a minimum area in changing its direction by a given amount, θ, it suffices to slide it along this curve in such a way as to be always tangent to it at the mid-point C, the motion to continue until a final position $A'B'$ has been reached whose direction makes the angle θ with the initial direction. In order to see this, we need only note that for any such movement of a line-segment the instantaneous center always lies on the perpendicular bisector.*

4. *The Original Problem.* Returning to the original problem of § 1, we see that it concerns a rotation of the segment through the special angle 180° with the further restriction that it shall finally return completely upon itself. In order for this to be accomplished by a movement which shall at all times have its instantaneous center upon the perpendicular bisector of the segment and hence, in accordance with § 3, shall sweep *over* a minimum area, it evidently suffices to give the segment a simple rotation of 180° about its own middle point, regarded as fixed. This, however, is not the only

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* As noted earlier, we are supposing, as the problem implies, that during the motion the direction of the segment changes only monotonically. It is for this reason that the single condition stated above concerning the *shape* of the curve is necessary.

possible solution, though it is the simplest. There are, in fact, an infinite number of other ways of producing the desired result. For example, first give the segment a pure translation by sliding it lengthwise any given distance along the indefinitely long line of which it forms a part, then give it a pure rotation of 180° about its middle point, then slide it back along the same line as before until it takes the desired position upon itself. For all such methods the instantaneous center always lies on the perpendicular bisector, this point being at infinity in the case of movements of pure translation.

5. *Problem of § 3.* We now proceed to consider the problem of § 3, and eventually that of § 1, when area swept *out* instead of swept *over* is to be minimized. The simplest case which can then arise is that in which, during the movement, no area is passed over more than twice; the next simplest case is that in which no area is passed over more than three times; next the case in which none is passed over more than four times; etc. Let us take for the moment the most general case; namely, that in which a certain area is passed over n times, but none more than this number of times. If, then, we let T represent the area swept *over* as the segment changes its direction by the amount θ, and let S_1 be the area passed over twice (duplicated), S_2 the area passed over three times (triplicated), $\cdots$, S_{n-1} the area passed over n times, we shall have as an equation for determining the area Σ swept *out:*

$$(1) \qquad \Sigma = T - S_1 - 2S_2 - 3S_3 - \cdots - (n-1)S_{n-1}.$$

We now proceed to consider in detail the simplest case; namely, that in which only duplication is present. We have $S_2 = S_3 = \cdots = S_{n-1} = 0$, so that (1) reduces to

$$(2) \qquad \Sigma = T - S_1.$$

Moreover, the greatest value which S_1 can take is $\frac{1}{2}T$, this corresponding to the extreme assumption that the entire area swept over is duplicated, as takes place for example when a segment is rotated through 360° about a fixed point in its perpendicular bisector. Thus, in addition to (2), we have

$$(3) \qquad S_1 \leqq \tfrac{1}{2}T.$$

It follows that any such movement must belong to one of the following four classes: (*a*) T not a minimum and $S_1 < \frac{1}{2}T$, (*b*) T not a minimum, but $S_1 = \frac{1}{2}T$, (*c*) T a minimum, but $S_1 < \frac{1}{2}T$, (*d*) T a minimum and $S_1 = \frac{1}{2}T$.

Of these it is easy to show that Σ can be a minimum only in case (*d*), it being assumed for the moment that there is a geometric possibility of a movement (or movements) in which (*d*) is realized. In fact, recalling from § 3 that when T is a minimum it has the value $l^2\theta$, we may write, corresponding to the four cases, the following:

$$
\begin{aligned}
&(a)\ \Sigma = T - S_1 > T - \tfrac{1}{2}T = \tfrac{1}{2}T > \tfrac{1}{2}l^2\theta,\\
&(b)\ \Sigma = T - S_1 = T - \tfrac{1}{2}T = \tfrac{1}{2}T > \tfrac{1}{2}l^2\theta,\\
&(c)\ \Sigma = T - S_1 > T - \tfrac{1}{2}T = \tfrac{1}{2}T = \tfrac{1}{2}l^2\theta,\\
&(d)\ \Sigma = T - S_1 = T - \tfrac{1}{2}T = \tfrac{1}{2}T = \tfrac{1}{2}l^2\theta.
\end{aligned}
$$

Thus, it is possible for Σ to attain its smallest value only in case (*d*); that is, T itself must be a minimum and $S_1 = \frac{1}{2}T$. Moreover, this smallest value of Σ (if geometrically realizable) has the value $\frac{1}{2}l^2\theta$, which we shall hereafter refer to as the *absolute minimum* for duplication.

It only remains to consider whether movements of class (*d*) are actually possible, and for this let us refer again to Fig. 1. Here, as the segment is rotated from AB to $A'B'$ the area S is duplicated, while R and T are passed over but once. However, by taking the radius OC sufficiently large, the values of R and T may be brought as near to zero as we please, thus leaving only the duplicated area S. Since, by § 3, T is a minimum for all such movements, it appears that the conditions represented in case (*d*), while not actually realizable in Fig. 1, may be made as nearly so as we please by choosing a sufficiently large radius OC. By such a movement, therefore, the value of Σ may be brought as near as we please to its absolute minimum, $\frac{1}{2}l^2\theta$, though not to this actual value. Moreover, it appears from geometrical considerations that no other method of moving the segment will produce similar conditions, for if the sliding of the segment is not done along a circular arc (the mid-point C always remaining the point of tangency in order to have T a minimum) then not all the area corresponding to S will be duplicated. In fact, the area generated by CB will then be either too large or too small to be exactly covered later by the area generated by end CA in its forward movement.

We therefore conclude that the problem of § 3, when considered with reference to area swept *out,* and with the assumption that no area is passed over by the moving segment more than twice, has no solution; that is, there is no one method of movement that sweeps out a minimum area. Nevertheless, the area in question may be brought as near as we please to the value $\frac{1}{2}l^2\theta$, the absolute, though unattainable, minimum.

Before passing to the consideration of similar questions when triplication is allowable, it is of interest to apply the results just noted to the special problem of § 1. Here $\theta = 180°$ and we have the further condition that the segment is finally to rest upon itself. The area swept over in doing this may be brought as near as we please to the value $\frac{1}{2}\pi l^2$, thus conforming to the above general statement, as follows: Let AB (Fig. 1) be the initial position and choose $A'B'$ so that θ shall be arbitrarily near to 180° (though not equal to this amount). Moreover, suppose that AB and $A'B'$ as thus drawn are extended until they meet, say at the point P. Consider now the movement obtained by first rotating the segment in the manner indicated in Fig. 1 to the position $A'B'$, then sliding it lengthwise along $B'A'$ produced until the midpoint lies at P, then a rotation about P until the segment lies lengthwise upon AB produced, and finally a sliding back along AB thus produced until the segment lies completely upon its original position. Evidently, in accordance with the general results above obtained, the area thus swept over can, by choosing θ sufficiently near to 180° and OC sufficiently large, be brought arbitrarily near to the indicated amount, $\frac{1}{2}\pi l^2$.

6. *Triplication of Areas.* We pass on to analogous studies when triplication as well as duplication is allowed. Here, instead of (2) and (3), we have respectively

$$\Sigma = T - S_1 - 2S_2, \tag{4}$$

$$S_2 \leqq \tfrac{1}{3}T, \tag{5}$$

the equality sign in (5) corresponding to the extreme assumption that the entire area swept over is triplicated.

From (4) we have $\Sigma + S_1 + S_2 = T - S_2$. But $S_1 + S_2 \leqq \Sigma$. Hence, $2\Sigma \geqq T - S_2$, or

$$\Sigma \geqq \tfrac{1}{2}(T - S_2). \tag{6}$$

Corresponding to the four cases (*a*), (*b*), (*c*), (*d*) of § 5, we may now consider the following as representing all possibilities: (*a*) T not a minimum and $S_2 < \frac{1}{3}T$, (*b*) T not a minimum, but $S_2 = \frac{1}{3}T$, (*c*) T a minimum, but $S_2 < \frac{1}{3}T$, (*d*) T a minimum and $S_2 = \frac{1}{3}T$. These assumptions, when employed in (6), lead to the following results respectively:

$$
\begin{aligned}
&(a)\ \Sigma \geqq \tfrac{1}{2}(T - S_2) > \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T > \tfrac{1}{3}l^2\theta,\\
&(b)\ \Sigma \geqq \tfrac{1}{2}(T - S_2) = \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T > \tfrac{1}{3}l^2\theta,\\
&(c)\ \Sigma \geqq \tfrac{1}{2}(T - S_2) > \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T = \tfrac{1}{3}l^2\theta,\\
&(d)\ \Sigma \geqq \tfrac{1}{2}(T - S_2) = \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T = \tfrac{1}{3}l^2\theta.
\end{aligned}
$$

Thus, it is only in case (*d*) that Σ can attain as low a value as $\frac{1}{3}l^2\theta$, which value, corresponding to the procedure of § 5, we may now take as the absolute minimum for triplication.

The geometric interpretation of these results, however, presents more serious difficulties than in the analogous results for duplication, for, if T be a minimum, as (*d*) requires, it is not apparent that triplication can be present in any species of movement to such an extent as to cover the entire area swept over, as (*d*) likewise requires, nor does it appear that such conditions can be realized through any limiting form of movement analogous to that presented in the expanding circular ring already employed in the case of duplication. It may therefore well be (and it is the author's belief, though he cannot furnish a formal proof of the fact) that the value $\frac{1}{2}l^2\theta$, already met with as the lowest approachable value in the case of duplication, is likewise the lowest approachable value even when triplication is allowed, the only difference in the two cases being that this value may actually be attained in the latter case for one or more special values of θ.*

Finally, it may be observed that the values of the so-called absolute minima for the cases where area may be passed over four, five, six, $\cdots$ times are respectively $\frac{1}{4}l^2\theta$, $\frac{1}{5}l^2\theta$, $\frac{1}{6}l^2\theta$, $\cdots$. The consideration of these cases, however, on the geometrical side again presents serious difficulties, but tends to the opinion, as in the case of triplication, that *in general* the smallest area that can be swept over by any actual movement of angle θ is $\frac{1}{2}l^2\theta$ rather than any of these smaller values.

UNIVERSITY OF MICHIGAN.

CONVERGENCE OF SEQUENCES OF LINEAR OPERATIONS†

BY T. H. HILDEBRANDT.

Let U_n be a sequence of linear continuous operations on the class F of functions f, continuous on the interval (a, b), i.e., suppose that every U satisfies the two conditions:

$$(1) \qquad U(c_1f_1 + c_2f_2) = c_1U(f_1) + c_2U(f_2)$$

for every pair of constants (c_1, c_2) and every pair of functions (f_1, f_2) of the class F;

(2) There exists a constant M depending on U such that if Nf is the maximum value of $|f|$ on (a, b) then

$$|U(f)| \leqq MNf.$$

The greatest lower bound of all possible values M might be called the *modulus* of U.

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* Thus, in case $\theta = \pi$ and triplication is allowed, the corresponding value $\frac{1}{2}l^2\pi$ may be attained as follows: Construct the hypocycloid of three cusps obtained by rolling the circle of radius $\frac{1}{2}l$ within the circle of radius $\frac{3}{2}l$ and let the given segment (of length $2l$) move so as to be always tangent to this curve and yet be everywhere entirely within it. The resulting area swept over as θ passes from 0 to π is entirely triplicated, as is well known, and is equal to the amount above stated, $\frac{1}{2}l^2\pi$. See, for example, F. Gomes Teixeira, *Traité des Courbes Spéciales Remarquables Planes et Gauches*, vol. II, p. 193. (Coïmbre, 1909.)

† Presented to the Society, September 4, 1919.

Helly[*] has shown that a necessary condition that $\lim_n U_n$ exists for every f of F is that the U_n be uniformly bounded, i.e., that there exist a constant M independent of n such that

$$|U_n(f)| \leqq MNf.$$

Then

$$\lim_n U_n(f) = U(f),$$

where $U(f)$ is a linear continuous operation whose modulus is less than or equal to M. In order to obtain sufficient conditions, it is necessary to make use of the classical theorem of Riesz, that every linear continuous operation on the class F is expressible in the form of the Stieltjes integral, $\int f d\alpha$, where α is of bounded variation. If α is regular, i.e., such that at every point x on (a, b) $\alpha(x)$ lies between $\alpha(x-0)$ and $\alpha(x+0)$, which can always be assumed to be the case without changing the value of the integral (and we shall restrict ourselves to this case), then the total variation $(\int |d\alpha|)$ of α is exactly the modulus of U, so that Helly's condition would be, that there exists a constant M independent of n such that

$$\int |d\alpha_n| < M,$$

for every n, i.e., the α_n are *uniformly of bounded variation.*

This condition is not sufficient. Helly[†] has shown that if (1) the α_n are uniformly of bounded variation and (2) there exists a function α of bounded variation such that

$$\lim_n \alpha_n = \alpha \text{ for every } x,$$

then

$$\lim_n \int f d\alpha_n = \int f d\alpha \text{ for every } f \text{ of } F.$$

Bray[‡] has shown that a weaker second condition is that there exists a function of bounded variation α and a denumerable everywhere dense set of points: $x_1, \cdots, x_m, \cdots$ including a and b, such that for every x_m

$$\lim_n \alpha_n(x_m) = \alpha(x_m).$$

This condition is not necessary. It is the purpose of this note to derive necessary and sufficient conditions.

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[*] Helly, Wiener Sitzungsberichte, vol. 121 (IIa) (1912), p. 268.

[†] Loc. cit., p. 288.

[‡] Annals of Mathematics, (2), vol. 20 (1919), p. 180.

We derive first an additional necessary condition. Since the α_n are necessarily uniformly of bounded variation, it follows from a theorem by Helly* that the α_n are a compact set, i.e., there exists a subsequence α_{n_m} of the sequence α_n and a function α (necessarily of bounded variation) such that

$$\lim_m \alpha_{n_m} = \alpha \text{ for every } x.$$

Consequently

$$\lim_m \int f d\alpha_{n_m} = \int f d\alpha$$

for every f of F. Since $\lim_n \int f d\alpha_n$ exists, it follows that

$$\lim_n \int f d\alpha_n = \int f d\alpha, \qquad \text{or} \qquad \lim_n \int f d(\alpha_n - \alpha) = 0$$

for every f of F. Let

$$\beta_n(x) = \alpha_n(x) - \alpha(x) - \alpha_n(a) + \alpha(a).$$

Then our condition becomes:

$$\lim_n \int f d\beta_n = 0$$

for every f of F. If we take $f = 1$ then

$$(a) \qquad \lim_n [\beta_n(b) - \beta_n(a)] = \lim_n \beta_n(b) = 0.$$

Again take $f = x$ for $a \leqq x \leqq \xi$ and $f = \xi$ for $\xi \leqq x \leqq b$. Then if we apply the integration by parts formula valid for Stieltjes integrals:

$$\int f d\beta_n = f\beta_n \Big|_a^b - \int \beta_n df = \xi\beta_n(b) - \int_a^\xi \beta_n dx.$$

Consequently, since $\lim_n \beta_n(b) = 0$, we must have

$$(b) \qquad \lim_n \int_a^x \beta_n dx = 0 \text{ for every } x \text{ of } (a, b)$$

or the equivalent condition:

(b') $\lim_n \int_{a_1}^{b_1} \beta_n dx = 0$ for every subinterval (a_1, b_1) of (a, b).

We transform this last condition as follows:

LEMMA I. *If β_n is a sequence of functions uniformly of bounded variation such that*

$$\lim_n \int_{a_1}^{b_1} \beta_n dx = 0$$

for every subinterval (a_1, b_1) of (a, b), and if u_n is the greatest lower bound of $|\beta_n(x)|$ for x on (a, b), then $\lim_n u_n = 0$.

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* Loc. cit., p. 283. See also Radon: WIENER SITZUNGSBERICHTE, vol. 122 (IIa), p. 1377, and Fischer, this BULLETIN, vol. 27 (1920), p. 12.

If $\lim_n u_n$ is not zero, then there will exist an $e > 0$ and a subsequence u_{n_m} of u_n such that $u_{n_m} > e$ for every n_m, i.e., for every n_m and every x we have

$$|\beta_{n_m}(x)| > e.$$

Since the β_{n_m} form a compact set, there exists a subsequence β_k' approaching a limiting function β also of bounded variation. This function β will obviously be such that $|\beta(x)| \geqq e$ for every x of (a, b). Consequently there exists a subinterval (a_1, b_1) of (a, b) such that either

$$\beta(x) \leqq - e \qquad \text{or} \qquad \beta(x) \geqq e$$

for every x of (a_1, b_1), i.e.,

$$\left|\int_{a_1}^{b_1} \beta(x)dx\right| > e(b_1 - a_1).$$

But the β_k' are uniformly bounded. Hence*

$$\lim_k \int_{a_1}^{b_1} \beta_k'(x)dx = \int_{a_1}^{b_1}\beta(x)dx,$$

which is not zero. Then we have reached a contradiction to the hypothesis of the lemma.

A direct consequence of this lemma is:

LEMMA II. *If β_n is a set of functions uniformly of bounded variation such that*

$$\lim_n \int_{a_1}^{b_1}\beta_n(x)dx = 0$$

for every subinterval (a_1, b_1) of (a, b), then in every subinterval (a_1, b_1) of (a, b) there exists a sequence of points x_n such that

$$\lim_n \beta_n(x_n) = 0.$$

For, any subinterval (a_1, b_1) of (a, b) may replace (a, b) in the hypothesis of Lemma I. Moreover, since $\lim_n u_n = 0$, it follows that there will exist a point x_n such that

$$|\beta_n(x_n)| \leqq u_n + \frac{1}{n},$$

i.e., $\lim_n \beta_n(x_n) = 0$.

The conclusion of this lemma together with the fact that the α_n are uniformly of bounded variation is also sufficient for the convergence under consideration, i.e., we have:

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* Cf. Lebesgue, *Leçons sur l'Intégration*, p. 114.

THEOREM. *Necessary and sufficient conditions that the limit* $\lim_n \int_a^b f d\alpha_n$ *exist for every* f *of* F *are that*

(1) *the* α_n *be uniformly of bounded variation,*

(2) *there exist a function* α *of bounded variation such that if*

$$\beta_n(x) = \alpha_n(x) - \alpha_n(a) - \alpha(x) + \alpha(a),$$

then

(*a*) $\lim_n \beta_n(b) = 0,$

(*b*) *in every subinterval* (a_1, b_1) *of* (a, b) *there exists a set of points* $x_1, \cdots x_n, \cdots$ *such that*

$$\lim_n \beta_n(x_n) = 0.$$

The function α of the theorem may be taken to be the function which is the limit of a subsequence of $\alpha_n(x)$.

To prove the sufficiency, we show that, under the hypotheses of the theorem, $\lim_n \int f d\beta_n = 0$ for every function f of F. From a theorem of Bray* it follows that

$$\left|\int f d\beta_n - \sum_{i=1}^{m} f(\xi_i)[\beta_n(x_{i-1}) - \beta_n(x_i)]\right| \leqq O_\delta M,$$

where $x_0 = a, x_1, \cdots, x_{m-1}, x_m = b$ is a subdivision of (a, b), ξ_i lies in the interval (x_{i-1}, x_i), O_δ is the maximum oscillation of $f(x)$ in (x_{i-1}, x_i) and $\int |d\beta_n| \leqq M$ for every n. Since f is uniformly continuous in (a, b), for every e, there will exist a d_e such that if $|x_i - x_{i-1}| \leqq d_e$, then $O_\delta \leqq e/2M$. Take a subdivision of (a, b) by the points $\xi_1 = a, \xi_2, \cdots, \xi_m = b$ such that for every i, $|\xi_i - \xi_{i-1}| \leqq \frac{1}{2} d_e$. Let $4d_0$ be less than the minimum of $\xi_i - \xi_{i-1}$. We apply the hypothesis of the theorem to the intervals $(\xi_i + d_0, \xi_{i+1} - d_0)$, i.e., select a set of points $x_{i,n}$ in each interval such that $\lim_n \beta_n(x_{i,n}) = 0$. We take $x_{0,n} = a, x_{1,n}, \cdots, x_{m,n} = b$, as points of division. Then since these points are finite in number, we can find an n_e such that if $n \geqq n_e$

$$\left|\sum_{i=1}^{m} f(\xi_i)[\beta_n(x_{i,n}) - \beta_n(x_{i-1,n})]\right| \leqq e/2.$$

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* Loc. cit., p. 179.

Since we also have

$$\left| \int f d\beta_n - \sum_{i=1}^{m} f(\xi_i)[\beta_n(x_{i,n}) - \beta_n(x_{i-1,\,n})] \right| \leqq e/2,$$

we have for $n \geqq n_e$

$$\left| \int f d\beta_n \right| \leqq e \qquad \text{or} \qquad \lim_n \int f d\beta_n = 0.$$

The conditions of the theorem may be simplified, if we note that the converse of Lemma II holds in the following form.

LEMMA II′. *If* β_n *is any set of functions uniformly of bounded variation, then a sufficient condition that* $\lim_n \int_a^x \beta_n dx = 0$ *for every* x *of* (a, b) *is that in every subinterval* (a_1, b_1) *of* (a, b) *there exist a sequence of points* $x_1, \cdots, x_n, \cdots$ *such that* $\lim_n \beta_n(x_n) = 0$.

The proof of this can be made along the lines of the sufficiency proof above.

As a consequence our theorem may be stated as follows.

Necessary and sufficient conditions that $\lim_n \int f d\alpha_n$ *exist for every* f *of* F *are that:*

(1) *the* α_n *be uniformly of bounded variation;*

(2) *there exist a function* α *of bounded variation such that if* $\beta_n(x) = \alpha_n(x) - \alpha_n(a) - \alpha(x) + \alpha(a)$, *then*

$$\lim_n \beta_n(b) = 0, \qquad \textit{and} \qquad \lim_n \int_a^x \beta_n dx = 0 \textit{ for every } x.$$

In this form it contains the following theorem of Lebesgue* as a special case. Necessary and sufficient conditions that $\lim_n \int f \varphi_n = 0$ for every f of F, the φ_n being summable on (a, b), are:

(1) $\int |\varphi_n| dx$ be bounded as to n;

(2) (*a*) $\lim_n \int_a^b \varphi_n \, dx = 0$;

(*b*) for every y, $\lim_n \int_y^b (y - x)\varphi_n(x) dx = 0$.

This results from the above theorem if we put

$$\beta_n(x) = \int_a^x \varphi_n(x) dx.$$

THE UNIVERSITY OF MICHIGAN.

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* *Sur les intégrales singuliers*, ANNALES DE TOULOUSE, (3), vol. 1 (1909), p. 57.

DE LA VALLÉE POUSSIN ON APPROXIMATIONS

Leçons sur l'Approximation des Fonctions d'une Variable Réelle. By C. de la Vallée Poussin. Paris, Gauthier-Villars, 1919. vi + 150 pp.

In 1908, M. de la Vallée Poussin was engaged in a study of certain questions in the theory of approximate representation by means of polynomials and finite trigonometric sums. In discussing the representation of a function whose graph is a broken line, he was led by the limitations of the formulas which he employed to make the following observation:*

"Il serait très intéressant de savoir s'il est impossible de représenter l'ordonnée d'une ligne polygonale avec une approximation d'ordre supérieur à 1 : n par un polynôme de degré n."

This sentence, occurring incidentally, with no particular emphasis, in a footnote attached to some supplementary pages following a memoir on a different phase of the subject, has been the direct or indirect occasion of about thirty published articles and memoirs which I call to mind at the moment,† and probably of numerous others which would be disclosed by a thorough search of the literature. The resulting theory is concerned not only with inner limits of approximation, as contemplated in the passage just quoted, but also with outer limits for the approximation attainable by various means, the degree of convergence of Fourier's and other series, and a variety of related topics. While this theory also has other origins and beginnings,‡ it is a fact that the numerous papers referred to can in each case be traced back in the personal experience and associations of the authors at least partly to de la Vallée Poussin's formulation of the problem.

The book under review, a monograph in the Borel series, is a summary, not of the entire literature of the subject, for citations are few and informal, but of its principal results, in systematic and often novel presentation. It is particularly appropriate that the man to whom the theory chiefly owes its inception, a man who has made essential contributions to it at various stages of its development, should now have performed the service of setting it before the general reader in its most attractive form.

Of the contents of the book in detail I shall speak more briefly than would otherwise be desirable, for the reason that I had occasion to refer to it extensively, as well as to the other literature, in an expository paper recently published in this BULLETIN.§

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* de la Vallée Poussin, *Note sur l'approximation par un polynôme d'une fonction dont la dérivée est à variation bornée*, BULLETINS DE L'ACADÉMIE DE BELGIQUE, CLASSE DES SCIENCES, 1908, pp. 403–410; p. 403, footnote.

† It will be a convenience to the reviewer in the present connection if he may be allowed the occasional use of the pronoun in the first person.

‡ See particularly Lebesgue, *Sur la représentation approchée des fonctions*, RENDICONTI DI PALERMO, vol. 26 (1908), pp. 325–328.

§ D. Jackson, *The general theory of approximation by polynomials and trigonometric sums. Chicago Symposium paper.* March 25, 1921. This BULLETIN, vol. 27, Nos. 9–10, June–July, 1921, pp. 415–431.

An Introduction deals with Weierstrass's theorems on the uniform approximation of continuous functions by means of polynomials and trigonometric sums, and with simple properties of the "modulus of continuity" of a function.* From certain references (later in the book) to passages in which I have made use of the latter, the reader might obtain the impression that it was, in the words of the White Knight, "my own invention." As I have said on occasion elsewhere, however, it came to me from the work of Lebesgue,† who used it in essentially the same way. I do not know what its earlier history may have been, if any.

Chapters I–III are concerned with the approximate representation of a given function by the partial sums of its Fourier series, the Fejér means of the Fourier series, and special trigonometric sums designed to give more rapid convergence in certain cases. In the main, the discussion is aimed at the determination of outer limits for the error of the approximation; an important part of the second chapter, however, is devoted to the theorems of S. Bernstein on the inner limit of the approximation of $|x|$ by a polynomial and on the derivative of a trigonometric sum.‡ Among the notable features of the treatment may be mentioned, in the case of the Fourier series, the simultaneous consideration of the original series and its conjugate, and in the case of the Fejér mean, the representation of the mean by the formula

$$\sigma_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(x + \frac{2t}{n}\right)\frac{\sin^2 t}{t^2}\,dt.$$

A generalization of the same formula is used for the purposes of the third chapter. In consideration of the acknowledgment made in the preceding paragraph, I may perhaps be allowed to say, in connection with certain comments on my thesis (e.g., pp. 43, 52), that its methods were considerably improved and its results extended in papers of mine published subsequently in the TRANSACTIONS.§

The author's methods are particularly convenient for the determination of the numerical constants contained in some of the error formulas. Take, for example, the theorem that if a function satisfies a Lipschitz condition with coefficient λ it can be represented by a trigonometric sum of order n with a maximum error not exceeding

$$\frac{K\lambda}{n},$$

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* By the *modulus of continuity*, or *modulus of oscillation*, $\omega(\delta)$, of a function $f(x)$, is meant the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$.

† Lebesgue, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz*, BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 38 (1910), pp. 184–210; see, e.g., p. 202. Also, Lebesgue, *Sur les intégrales singulières*, ANNALES DE TOULOUSE, (3), vol. 1 (1909), pp. 25–117; p. 114 and elsewhere.

‡ Cf. the Symposium paper cited above.

§ *On approximation by trigonometric sums and polynomials*, TRANSACTIONS OF THIS SOCIETY, vol. 13 (1912), pp. 491–515; *On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series*, TRANSACTIONS, vol. 14 (1913), pp. 343–364.

where K is an absolute constant. In one of the papers just referred to, it was shown that K can have the value 2.90, but certainly not a value less than $\pi/2$. Gronwall* later justified the use of the value $K = 2.76$. The author's calculation on page 45, applied to the case of a simple Lipschitz condition, and supplemented by the use of the relation†

$$\int_0^\infty \frac{\sin^4 t}{t^3}\, dt = \log 2,$$

gives at once

$$K = \frac{12 \log 2}{\pi} = 2.65.$$

Dr. Gronwall has suggested to me in correspondence a method by which I believe it would be possible to reduce the value considerably further, but I have not carried through the computation. From page 46 of the text it can be deduced readily that if $f(x)$ has a first derivative satisfying a Lipschitz condition with coefficient λ, it is possible to obtain an approximation with an error not exceeding $6\lambda/n^2$. The constant 6 is an appreciable improvement over the value 9 (more closely, $2.90^2 = 8.41$, or, by inference from Gronwall's work, $2.76^2 = 7.62$) previously published.

The remaining chapters are no less significant, but will be dismissed very briefly. Chapter IV presents general theorems on the inner limit of approximation by trigonometric sums; the fifth chapter deals with outer and inner limits for polynomial approximation; the next two are devoted to the Tchebychef theory of approximation by polynomials and trigonometric sums, with some of its recent extensions. The last three chapters contain a study of the approximate representation of analytic functions of a complex variable.

A number of misprints have been noted, particularly in the early pages. For the most part they are not such as to cause any serious inconvenience. In the footnote on page 3, however, the reference should be to the BULLETIN DES SCIENCES MATHÉMATIQUES.

The book needs no higher praise than a statement that it is worthy of its author. It is qualified to give pleasant hours to any student of mathematics who turns its leaves, and will be of inestimable value to anyone who has occasion seriously to study the subject with which it deals. If many a mathematical treatise seems to place itself under the motto, quoted in a different connection not long ago by a reviewer in these pages, "All hope abandon ye who enter here," it would be appropriate to inscribe on the fly-leaf of a text by M. de la Vallée Poussin:

"The crooked shall be made straight, and the rough places plain."

DUNHAM JACKSON.

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* *On approximation by trigonometric sums*, this BULLETIN, vol. 21 (1914–15), pp. 9–14.

† I do not remember seeing a proof of this relation in print; I am personally indebted for various demonstrations of it to Messrs. Gronwall, Landau, M. Riesz, and I. Schur. See, however, Gronwall, loc. cit., p. 14.

SHORTER NOTICES

Notes d'Histoire des Mathématiques (*Antiquité et Moyen Âge*). By B. Lefebvre, S. J. Louvain, Societé Scientifique de Bruxelles, 1920. viii + 152 pp. Paper.

This little work is made up of a series of articles published under the title "Autour d'une Histoire des Mathématiques" in the REVUE DES QUESTIONS SCIENTIFIQUES during the years 1907–1911. These articles have been revised since their original appearance and are now published in the hope that they may be, as they will be, of further service to teachers and students.

The author is a member of an order that has produced many mathematicians of note, many devoted teachers, and a body of self-sacrificing men who, in the seventeenth century, carried Western mathematics to the Far East. Such men as Ceva, Cavalieri, Clavius, Matteo Ricci, Verbiest, and Vincenzo Riccati, to name only a few, have worthy successors, as teachers, in such Belgian scholars as Vanhée, Bosmans, and Lefebvre—men who, in the years of the holocaust, suffered great privations—Vanhée being imprisoned at hard labor, Lefebvre seeing the library at Louvain destroyed and his people killed, and Bosmans spending the years in relieving the destitute in Brussels. It is, therefore, a pleasure to see, in such a work as this, an evidence of the spirit to "carry on," and to bring Belgium back to a full appreciation of her standing as a home of scholars.

The work is divided into fourteen chapters, as follows: I. The historians of mathematics; II. Oriental mathematics; III. Mathematics of Rome, of Byzantium, and of the Arab schools; IV. Numeration of the Greek, Roman, and Medieval scholars; V. Hindu origin of our numerals; VI. The development of the Hindu-Arabic system; VII. Mathematics in the Middle Ages in general; VIII. The early Middle Ages; IX. The time of Charlemagne; X. Gerbert (Sylvester II); XI. The exact sciences in Belgium; XII. First Renaissance of mathematics in the twelfth and thirteenth centuries; XIII. The Arabs in Europe; XIV. The School of Toledo in the twelfth century. Such an array of titles is tempting to the general reader as well as to the teacher and the mathematician.

Père Lefebvre is modest in his claims. He frankly states that his work, say the equivalent two hundred pages of the more conventional size, is merely a set of notes; and yet a set of notes of this extent may be very valuable, or it may be useless, or even worse. It is needless to say, however, that the work of a man of Père Lefebvre's erudition can not be other than helpful to students of mathematics, and interesting and suggestive, to say the least, to scholars in general.

His first chapter, on the historians of mathematics, is essentially a critical review of Ball's well known work, but it contains a judicial estimate of the works of other historians, such as Montucla, Tannery, Chasles, Zeuthen, and Cantor, not to speak of such lesser and often inferior writers as Marie and Hoefer. As to Mr. Ball, the criticisms extend throughout the work, and are directed not only against the translation, which was carelessly made, but against the original text, especially for its lack of facsimiles to illustrate such matters as the ancient methods of writing equations, for its neglect of oriental mathematics, for its failure to make use of such researches as those of Tannery, and for such doubtful statements as those relating to the discovery of the conic sections by Menæchmus and the destruction of the Alexandrian library by the Christians.

The criticism that little or no attention is paid by Mr. Ball to the contributions of Boethius, Capella, Cassiodorus, and Varro is more easily answered, since the English historian had to limit his space, and on the whole he seems to have chosen fairly well. If he should wish to return Père Lefebvre's criticism in kind, he might well ask what he should have omitted in order to find space for such names as those mentioned. He might even go further and inquire by what right his critic should speak of "Mohammed-ben-Mouça Al-Hovarezmi, *surnommé* Al-Khorizmi," since "Al-Hovarezmi" and "Al-Khorizmi" are merely different transliterations of the same Arabic name,—the latter being the better of the two. Indeed, it may well be asked why the French writers have, in general, been so lacking in uniformity in their transliterations of all oriental names, a difficulty that foreign readers find very troublesome. Mr. Ball might also ask why Mohammed ibn Musa's name also appears as Al-Hovarez (p. 22), Al-Kharizmi (p. 121), and Al-Khowarez (p. 133), and whether these various forms are not unnecessarily confusing. He might also inquire as to the validity of the statement (p. 23), "Le surnom Al-Khorizmi devint le titre, *dans le langage courant des Arabes* (Italics mine), d'un ouvrage du même vieil auteur sur l'Arithmétique hindoue." This statement is, so far as this reviewer is aware, an unwarranted one, and it would be interesting to have it confirmed. The facts seem to be that the medieval Latin translators did their best to put the phrase "the book of al-Khowârizmi" into their tongue, and the best they could do was to write *Liber Algorismi*, or *Liber Algoritmi*, whence came our "algorism," "augrim," and other variants. Mr. Ball might also ask the authority for such a statement as that the Greeks (presumably, of course, those of the classical period) knew the proof of nines, and that it may have been known to the Pythagoreans. If any unquestionable authority for these statements exists, this reviewer is not aware of it.

On the other hand, Père Lefebvre has given his readers much that will be very helpful, as in his brief but scholarly treatment of the Roman, Greek, and Hindu-Arabic numerals, in his notes on the French abacus, and particularly in his statements as to the debt we owe to the Christian Church in the Dark Ages. We have no better essay upon the contributions of the medieval monks to the preservation if not the advance of mathematics in the centuries from the ninth to the thirteenth than Père Lefebvre has here set forth. This might naturally be expected, for a churchman is writing of other churchmen, and he does so not merely with a desire to place their work in the proper light, but with a knowledge, derived from an access to material, that few others possess. It is here, indeed, that the work assumes a value all its own, and one which renders it a necessity in the library of the student of the history of mathematics.

Père Lefebvre's work shows the effect of writing and publishing his essays at different times. There is a lack of that perfect coordination which comes from uninterrupted labor. Moreover, the work, as has been shown above in a few instances, has various slips of the pen or of the memory, little errors, in the main, that it is quite unnecessary to mention in a brief review. The treatment of the Middle Ages is so scholarly and helpful as far to outweigh the minor imperfections that, considering the tumult of recent years, can easily be overlooked. Rather than search for errors of no moment, we should express our indebtedness to one who, with all his cares in these troublous times, has collected his material and made it more available for historians and students of mathematics.

DAVID EUGENE SMITH.

Lezioni di Meccanica Razionale, Seconda Edizione. By Pietro Burgatti. Bologna, Nicola Zanichelli, 1919. xi + 544 pp.

The first edition of this work was published in 1916. Although the present edition contains about fifty pages more than the first, the topics treated and the method of treatment remain unchanged; the additional pages being due to brief additions scattered throughout the various chapters. There are no exercises, but it is stated in the preface that the author intends to publish a separate book of exercises with solutions.

The book begins with a chapter on vector analysis containing as much of the subject as is needed for the development of the mechanics of a rigid body, which is taken up in the immediately following chapters. This development, aside from the fact that vector methods are used, is pretty much along traditional lines, i.e., kinematics of a particle and of a rigid body, including some discussion of the geometry of motion; statics, in which problems are first solved by writing the equations of equilibrium and later by the method of virtual work; dynamics, which includes the theory of the top, generalized coordinates, and a brief chapter on the problem of three bodies. Before taking up the last two chapters which give a brief introduction to the mechanics of deformable bodies, the author finds it necessary to insert another chapter on vector analysis, in which are considered such matters as divergence and curl of a vector, the theorems of Green and Stokes, and Poisson's equation. The closing chapter deals with the historical development of mechanics.

It is recognized that the student of mathematical physics must be familiar with vector analysis, and texts on electricity and magnetism generally begin with a mathematical introduction which gives the machinery required for what is to follow. More recently the same idea is being extended to mechanics. We now have a number of books which treat the mechanics of a rigid body and also the mechanics of continua by vector methods, but the number is not so large but that the book under review can find a hearty welcome. So much of the subject of mechanics is essentially geometric that it is particularly well adapted for study by vector methods, and any careful teacher, whether he uses the formal vector notation or not, is sure to bring out the geometric interpretation. The thing that must be remembered, however, is that even when a property is expressed in vector notation, it may be necessary to add some explanation or to put things in a way that is not too abstract in order that the student meeting the expression for the first time may see clearly the significance of the matter under discussion, for vector methods should add not only brevity but clearness. The second chapter on vector analysis would seem less elegant but would be more easily understood by a student if more effort were made to bring out either geometric or physical interpretations.

The author has done well in showing no particular aversion to the cartesian coordinate system, as a familiarity with the various quantities in both the vector and cartesian notation is needed. In view of the statement in the preface that it was desired to show how to treat problems that arise either in science or in the field of technical applications, it is surprising that so little attention was paid to graphical statics. On the other hand it may of course be argued that the one thing in mechanics which the engineer is sure to be familiar with is graphical statics. The treatment in many cases is very elegant, as for instance in the derivation of the acceleration, the theorem of Coriolis, and Euler's equations.

The book is suited to our seniors and graduate students and could be covered in a three-hour course throughout the year.

PETER FIELD.

Mathematik des Geld- und Zahlungsverkehrs. By Alfred Loewy. Leipzig and Berlin, Teubner, 1920. viii + 273 pp.

This work is a very readable elementary treatise on the mathematics of finance with the usual chapters on simple interest, compound interest, annuities certain, amortization and sinking funds, with the accompanying compound interest and annuity tables. Life annuities and insurance problems are not discussed. The general treatment is not so formal as that of most of the English and American books on financial mathematics. Much historical and economic matter is included:—for example there is a very interesting historical account of the rate of interest from the earliest times and a clear explanation of the question of foreign exchange. The book had been sent to the publisher in 1918, but was withdrawn by the author and practically rewritten in order to bring it up to date for German readers by inserting discussions of the various new laws and taxes arising out of the financial condition of the country at the close of the war. At one point the author rather gloomily concludes a discussion of the low exchange value of the mark: "An einem Übergang zur Bareinlösung des jetzt in Deutschland valutarischen Papiergeldes durch Gold ist für lange Zeit nicht zu denken."

The notation of the Text-Book of the Institute of Actuaries is used in the book with a few trifling exceptions, the most noticeable being $j^{(m)}$ as an abbreviation for $m\{(1+i)^{1/m}-1\}$ instead of the usual notation $j_{(m)}$.

On the physical side there is much to be desired in this book. The typography is fairly good, but the matter is very much crowded on the narrow-margined page. Before the reviewer had finished reading, the book had changed into a mass of loose leaves of very poor paper.

A. R. CRATHORNE.

Bibliotheca Chemico-Mathematica. Compiled and annotated by H[einrich] Z[eitlinger] and H. C. S. London, Henry Sotheran and Co., 1921. 2 vols.: vol. I, xii + 428 pp.; vol. II, iv + 429–964 pp., numbered consecutively from vol. I; 127 plates.

This work was projected some time before the war, subscriptions being then invited, but it was not completed until the autumn of 1920, the publication taking place in the spring of 1921. It consists of a combination of catalogues of books in the large stock of Henry Sotheran and Co., of London, the well known dealers in standard treatises of various kinds, and contains more than 17,000 items. The catalogues were issued during a period of several years, and the prices represent conditions which have, in the course of events, been modified from time to time. As a sales catalogue of works on mathematics, astronomy, physics, chemistry, and allied subjects it is valuable not only because of the accuracy with which, in general, it has been compiled, but because it represents the trade conditions during a critical period of the world's history.

Of course such a work cannot, in the nature of things, be complete. This one represents simply the books that Messrs. Sotheran had in stock when the catalogues appeared. Since this stock was, however, unusually large, the work becomes a reference book of great value. This value is increased, in this instance, by the insertion of 127 plates containing 247 portraits and facsimiles, many of the latter being photographic reproductions of rare titlepages, and thus being of great help to the bibliographer.

At the close of the second volume there is an elaborate index of 96 pages, arranged by topics, a subject like geometry, for example, being subdivided into algebraic, analytic, cross-ratio, descriptive, differential, elementary, enumerative, Euclidean, four-dimensional, general, Greek, higher, infinitesimal, meta-, modern, natural, non-Euclidean, pictorial, plane, practical, projective, pure, solid, symbolic, and synthetic, with various further sub-topics of the second order.

Such an index would be of little use were it not for the fact that the catalogue is arranged alphabetically by names of authors, so that the index will not often be consulted. Even as it is, however, there is some difficulty in finding the items one wishes, owing to the fact that the work consists of a large catalogue, a supplement, a set of addenda, and several sub-catalogues. Suppose, for example, one who is unfamiliar with the work wishes to find the Scientific Papers of John Couch Adams, where would he look? The answer is, under "Further Addenda" on page 786; but what should lead him to look there is a mystery. Any reader who happens to have the work at hand might be interested to see how long it takes to run down Reuleaux's work on kinematics, published in 1876 (not the one of 1875). It is, therefore, a matter of regret that the publishers could not have given an index of authors. As between that and the one so laboriously compiled, the one by names would have been far more useful.

In spite of the annoyances that come from the index, however, the work is so valuable that it should have a place upon the shelves of all who are interested in the history or the bibliography of mathematics.

David Eugene Smith.

Kepler. By Walter W. Bryant. London, Society for Promoting Christian Knowledge (New York, Macmillan), 1920. 62 pp. + portrait.

This little work is one of the popular series of biographies now appearing under the name of "Pioneers of Progress," published by the Society for Promoting Christian Knowledge. It is written by a member of the staff of the Royal Observatory at Greenwich, one who knows astronomy, is conversant with its noble history, and appreciates the contributions of the greatest mathematical astronomer of his time, and perhaps of all time.

The date of the publication of the essay is appropriate, the current year (more precisely, December 21, 1921) being the three hundred and fiftieth anniversary of Kepler's birth. It is also timely that the world should consider the contributions which Kepler made to the theory of celestial mechanics, comparing them, and those made by Newton and Laplace, with the new theory which is now attracting so much serious attention on the part of scholars and attaining so much notoriety in the public press.

The work consists of six chapters, as follows: I. Astronomy before Kepler; II. Early life of Kepler; III. Tycho Brahe; IV. Kepler joins Tycho; V. Kepler's laws; VI. Closing years. To these are added a list of dates, a brief bibliography, and a glossary of terms that will be helpful.

The first chapter gives a brief sketch of the leading theories advanced by Greek and medieval scholars to account for the motion of the planets, and then sets forth the nature of the advance made by Copernicus. The author calls attention to the common error of believing that the heliocentric theory was clearly stated by Copernicus; for the latter, "still hampered by tradition, did not reach" it. Most of the work is, however, devoted to Kepler and his discoveries. These were, as is well known, intermingled with naive theories that can well be excused in the childhood of modern astronomy. The supposed relation of the five planets (as then known) to the five regular polyhedrons is Pythagoreanism two thousand years delayed, and it seems inconceivable to us that the one who discovered Kepler's laws could have advocated such an absurd theory. The fact simply helps us to appreciate the intellectual atmosphere in which the continental scientists of Shakespeare's time were living.

Kepler's association with Tycho Brahe is briefly set forth. The latter was well advanced in the study of astronomy before Kepler was born, and when the latter was less than a year old Tycho was making observations of considerable importance. Indeed, the celebrated observatory at Uranienborg was built when Kepler was only five years of age, and here and then Tycho was prescribing sulphur to cure infectious diseases "brought on by the sulphurous vapors of the Aurora Borealis"! It was not at this great observatory, however, that Kepler took up his studies, but in Prag, whither Tycho had gone after the death of his Danish patron. It was here that the younger man began his great work, the details of which are skillfully summarized in the essay under review. The mingling of the mystic and the scientist, of the astrologer and the astronomer, of the small mind and the giant intellect are so well set forth in a few pages that the little work may safely be recommended as one of the best of the series. It should be welcome in the schools and is justified in the library of the scholar.

DAVID EUGENE SMITH.

Darstellende Geometrie. Vol. II. By Theodor Schmid. Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1921. (Sammlung Schubert, XVI.) 315 pp., 156 figures.

The first volume of this work appeared in 1912, and was reviewed in this BULLETIN, vol. 21 (1914), pp. 204–205. A second edition appeared in 1919, containing slight changes (mentioned in the BULLETIN, vol. 27 (1921), p. 285). The second volume preserves the same general arrangement; the text is frequently interrupted by lists of unsolved exercises, and a useful historical and bibliographical note is added at the end of each chapter. The figures are well drawn and are usually clear, but in some of the more complicated ones, as numbers 107, 112, 118, 124, 126, the wealth of detail makes them difficult to follow.

The book begins with cavalier perspective, or oblique axonometry, and a shorter treatment of orthogonal axonometry. Besides a fairly full treatment of polyhedral figures, a shorter one on circles, spheres, cylinders and cones is added. The proofs are largely geometric, but footnotes use methods of analytic geometry and the calculus. The second chapter, that on central perspective, is particularly well written. The reader is introduced to the various steps and purposes of this method in a very attractive and easy manner. It might be hinted that the author became somewhat ambitious in treating singularities of plane and space curves, but there was a real temptation to complete the argument, and after all, the more complicated formulas appear only in the footnotes. In other respects the development is elementary. A wealth of important applications, including theater perspective, are treated in the sixty pages of this chapter.

The chapter on surfaces of revolution and on tubular surfaces is necessarily more mathematical. Contour, tangent planes, normal planes, parallel sections and principal meridians are treated. The application to quadrics of revolution is first to obtain the proper algebraic result, and then to interpret it graphically. But when shades and shadows are considered, the apparent contours from oblique infinite sources, or from a finite center, are too complicated to be interpreted from the equations, and here the method of the preceding chapter is extensively employed. In the discussion of cyclides of revolution both the algebraic and graphic methods of central perspective are employed. Later, more general tubular surfaces, together with their intersections, and their shadows, are considered.

The fourth chapter, of eighty-three pages, is devoted to helicoids and to non-developable ruled surfaces. It begins with an algebraic discussion of the simple helix, then of the ordinary helicoid, followed by certain graphic representations, such as tangent, indicatrix, and the intersection with a cone of revolution, each result being first derived analytically and then interpreted graphically. After a short discussion of parallel illumination, the more complicated problem of oblique generators is taken up, the surface generated being a quasi helicoid of high pitch. To this surface the same questions are applied as to the ordinary helicoid, and the results compared. Then comes the drawing of cylindrical and conical screws, with contours and plane projections. Finally, a number of particular ruled surfaces are defined and described, the most attention being given to the cubic conoid. The intricate problems treated in this chapter are much simplified by the consistent use of the same method throughout; first an algebraic discussion, and then the graphical interpretation of the results obtained.

The last chapter contains a brief introduction to topological mapping, to the intersections of roofs, and to the principles of the Mercator map.

The press-work is good, and the proof-reading excellent. The book is not provided with an index.

VIRGIL SNYDER.

Die Quantentheorie, ihr Ursprung und ihre Entwicklung. By FRITZ REICHE. Berlin, Julius Springer, 1921. vi + 231 pages.

The theory of relativity and quantum theory are probably the greatest philosophical developments of modern physics. Both grew out of the failure of classical mechanics to give correct results when applied to radiant energy. Neither can be regarded as rigorously established on physical fact. Both are rather working hypotheses that have not been found inconsistent with fact. Of the two the theory of relativity is much more completely formulated. In fact there is in the minds of most physicists still considerable uncertainty as to just what quantum theory is, and concerning its cause views differ all the way from those of the extreme atomists, who consider radiant energy of a given frequency as capable of existing only in pieces of a certain size, to those of the older exponents of continuity, who still believe these curious results due to averaging. There is also the fascinating suggestion that we have here a consequence of sub-electronic structure and so have made a first step into the mysterious region beyond the electron. Because of all this uncertainty the subject has for the investigator an interest transcending that of more highly crystallized theories. For this reason we particularly welcome a book like that of Reiche which presents in simple form the theory and its principal applications.

The author disclaims any intention of writing a systematic textbook, yet he has produced as systematic a text as exists on the subject, and a very readable one. The first three chapters give Planck's hypothesis of energy quanta assumed in order to obtain a radiation formula agreeing with experiment, Einstein's hypothesis of light quanta with photo-electric application, and Planck and Sommerfeld's hypothesis of quanta of action. The remaining chapters treat the Einstein-Debye theory of specific heat of solids, the specific heat of gases, the Bohr type of atom with application to optical series, X-ray spectra, and some molecular models. The book should not be used as a substitute for Planck's *Heat Radiation* or Sommerfeld's magnificent *Atombau und Spektrallinien*, but as an introduction to the subject with which one may physically orient himself before taking up more complex discussions such as occur, for example, in the recent pamphlet of E. P. Adams published by the National Research Council.

H. B. PHILLIPS.

Tavole di Numeri Primi entro Limite Diversi, e Tavole Affini. By L. Poletti. Milan, Manuali Hoepli, 1920. xx + 294.

This little volume will be welcomed by many workers in the theory of numbers to whom the larger factor tables and lists of primes are not available. Even for those who are well equipped with such tables, certain features of Poletti's book will be found of value and interest.

Following a dedication to Gino Loria, which is quite poetic in its enthusiasm, there is a preface by Loria himself which gives a short account of the history of such tables. Then follows a series of tables that cover some 100 pages and give lists of primes between various limits. Table I lists the primes in the first 200,000 numbers with the exception of unity. Table II lists the primes in the first 100,000 successive numbers beyond 10,000,000. Table III lists those in the first 10,000 successive numbers beyond 100,000,000. Table IV lists those in the first 100,000 successive numbers beyond 1,000,000,000. Besides these tables there are six short tables giving the primes in the following ranges:

32,258,101 to 32,261,279,	52,631,591 to 52,636,823,
34,482,761 to 34,486,201,	58,823,533 to 58,829,411,
43,478,269, to 43,482,599,	76,923,101 to 76,930,757.

It is worth while to have these results available even though they are but tiny islands in the vast sea of the unknown, as anyone will admit cheerfully who has ever been faced with the task of determining the character of a number of such an order of magnitude.

Table V gives all the factors of all numbers less than 50,000 exclusive of multiples of 2, 3 and 5. It is arranged in the usual fashion with respect to the modulus 30, so that the number $30x + y$ will have all its factors in column headed y and in line number x. The author, for some reason which may produce some confusion, numbers the lines on the right so that the factors of $30x + y$ are found in line number $x + 1$. The scheme of listing is by no means as convenient for reference as that employed by Burckhardt, but it enables the author to use a much smaller page.

Table VI gives a scheme for finding the product of any two numbers less than 50,000 which are prime to 2, 3 and 5, being a set of tables for obtaining that line of table V where that product is to be found. The scheme is an ingenious one, but the practical importance for the computer is doubtful.

Tables XIII to XVII give the number of primes between certain limits, some of the tables distinguishing between primes of various linear forms

Table XVIII gives the list of primes of the form $n^2 - n - 1$ which lie between 1 and 10,400,000. The series of numbers given by this form is an example of what the author calls an Eratosthenean series, the sifting process being applicable to it to determine the primes in it. The author cites Eratosthenean series of higher orders, but does not indicate what restrictions should be placed on the forms to insure Eratosthenean properties, beyond the restriction that the form be irreducible. Thus the series $x^3 - 3x - 1$ is Eratosthenean, while the series $x^3 - 3x - 5$ is not. We are not told how to determine when the sifting process is applicable and when it is not. The author shows that every form of the second order gives an Eratosthenean series, and he gives in effect, by a laborious and not very convincing method of proof, the theorem that a quadratic congruence has, for a prime modulus, two roots or none. He does not seem to see the connection between the numbers represented by the form $ax^2 + bx + c$ and those represented by the binary quadratic form (a, b, c).

The author is disappointing in his answer to the question "What confidence can be placed in the accuracy of these tables?" Where accuracy is of such vital importance the user of such a table is entitled to know of some of the checks and controls employed, and if comparison has been made with existing tables, what safeguards have been employed in making the comparison. Until some other independent computation has been made of the lists of primes here given, it will be well to use them with due caution.

D. N. LEHMER.

Meccanica Razionale. By C. Burali-Forti and I. Boggio. Turin and Genoa, S. Lattes and Co., 1921. xxiv + 425 pp.

Students of mathematics who are familiar with the chapter on the application of vector analysis to mechanics in Burali-Forti and Marcolongo's book on vector analysis will be particularly interested in the book under review, since it is much more easy to find topics in mechanics which can be conveniently treated by vector methods than it is to treat the whole subject by such methods. It is stated in the preface that the authors do not present a complete treatise on rational mechanics, but that they give certain general notions which form a necessary foundation for applied mechanics. The latter part of this statement is possibly misleading, as the authors do give a systematic introduction to mechanics, using vector methods, and cover ground not much different from what is ordinarily offered in courses on mechanics in our American universities. The principal differences are that the book under review devotes more attention to geometry of motion, contains no sets of exercises, and does not take up the dynamics of the top. The pages are small, about $4\frac{1}{2}''$ by $6\frac{1}{4}''$, but to those who are familiar with the work of the authors, it is needless to state that they possess the ability to treat a subject adequately in a minimum amount of space.

The knowledge of vector analyiis and of homographies assumed on the part of the reader is briefly outlined in a thirty-eight-page introduction. As most of our students study mechanics before they have taken a course in vector analysis, an instructor using this book as a text will find it necessary to spend some time clarifying the introduction. The book will be found very suggestive also in connection with a course on vector analysis.

PETER FIELD.

NOTES

The two hundred twenty-second regular meeting of the American Mathematical Society, being its seventeenth regular meeting in the West and the forty-ninth meeting of the Chicago Section, will be held on Friday and Saturday, April 14 and 15, 1922, at the University of Chicago, in honor of the twenty-fifth anniversary of the Chicago Section. On Friday afternoon, Professor A. B. Coble will give a symposium lecture on *Cremona transformations and applications to algebra, geometry, and modular functions.*

The February meeting of the Society was held in New York at Columbia University on February 25. By request of the programme committee Professor J. L. Coolidge gave an address on *The basis of mathematical probability.*

The Council has accepted an offer from an anonymous donor to pay the cost, up to four thousand dollars, of an extra volume of the TRANSACTIONS OF THIS SOCIETY, to be brought out promptly. This extra volume will be sent without charge to all subscribers and exchanges now on the list.

The July number (vol. 43, no. 3) of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *Integral products and probability*, by P. J. Daniell; *Introduction to a general theory of elementary propositions*, by E. L. Post; *Note on Schläfli's elliptic modular equations*, by Arthur Berry; *Associated forms in the general theory of modular covariants*, by Olive C. Hazlett; *On* (2, 3) *compound involutions*, by T. R. Hollcroft.

The June number (ser. 2, vol. 22, no. 4) of the ANNALS OF MATHEMATICS contains: *An analytical solution of Biot's problem*, by Tsuruichi Hayashi; *Minimal surfaces containing straight lines*, by J. K. Whittemore; *An extension of Green's lemma to the case of a rectifiable boundary*, by E. B. Van Vleck; *Periodic conjugate nets*, by E. S. Hammond; *On the transformation of convex point sets*, by J. L. Walsh.

Professor H. A. Lorentz, of the University of Leyden, will lecture at the University of Wisconsin from March 20 to March 30, on *Light and the constitution of matter.* A colloquium in his honor on *Fundamental concepts of electrodynamics and of the electron theory* will be held under the auspices of the Department of Physics of the University of Wisconsin on March 30–31 and April 1.

The former students of the Department of Mathematics of Bryn Mawr College, through Dean Isabel Maddison, announce that a mathematical meeting will be held at Bryn Mawr on April 18 in honor of Professor C. A. Scott, on the completion of her thirty-seventh year as Head of the Department of Mathematics. After an address of welcome by President M. C. Thomas, and an introductory address by Miss Marian Reilly, Professor A. N. Whitehead, of the South Kensington Imperial College of Science will speak on *Relativity and gravitation, group tensors, and their application to the formulation of physical laws.* An honorary committee for this occasion consists of Professors R. C. Archibald, G. D. Birkhoff, E. W. Brown, F. N.

Cole, J. A. Eiesland, James Harkness, E. R. Hedrick, F. P. Lewis, Isabel Maddison, E. N. Martin, H. A. Merrill, E. H. Moore, Frank Morley, L. W. Reid, R. G. D. Richardson, E. J. Townsend, Oswald Veblen, H. S. White, and R. G. Wood.

The Massachusetts Institute of Technology has begun publication of a new periodical to be known as the JOURNAL OF MATHEMATICS AND PHYSICS OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY, of which the first number appeared in November, 1921. The editors are Professors F. S. Woods, H. M. Goodwin, F. G. Keyes, and C. L. E. Moore.

In September, 1921, the Helmholtz-Gesellschaft was formally organized at the meeting of physicists held at Jena. This Society has accumulated a considerable capital, the interest of which is to be used to support scientific work in physics and technology. In celebration of the one hundredth anniversary of the birth of Helmholtz, which occurred August 31, a *Festschrift* will be published, which will include a paper by Paul Hertz on the foundations of geometry.

On January 23, Professor David Hilbert, of the University of Goettingen, celebrated his sixtieth birthday. On this occasion, his friends, colleagues, and former students presented to him an address, an album of photographs, and a memorial volume of mathematical papers. Among those who joined in these remembrances were over sixty-five American friends and former students. The celebration was directed by a committee consisting of Professors O. Blumenthal (chairman), R. Courant, G. Hamel, E. Hecke, A. Schönflies, and (for America) E. R. Hedrick. The mathematical papers of the memorial volume will appear separately either in the MATHEMATISCHE ANNALEN or in the MATHEMATISCHE ZEITSCHRIFT.

It is announced that the preparation of the complete edition of the works of Sophus Lie, undertaken in 1912 by Teubner but suspended because of the greatly increased cost of printing, will be resumed with the financial support of the Norwegian Mathematical Society. The title of the edition will read: *Sophus Lie, Gesammelte Abhandlungen, im Auftrage des Norwegischen Mathematischen Vereins und mit Unterstützung der Akademien zu Kristiania und Leipzig herausgegeben von Friedrich Engel und Paul Heegaard.* It is planned to publish seven volumes, of which volume three, the first to appear, is now in press.

The mathematical library of the late Professor Paul Stäckel, of Heidelberg, is offered for sale by Walter Richters Buchhandlung und Antiquariat, at Leipzig, for 45,000 marks. It consists of more than 900 volumes.

The members of the Division of Mathematics of Harvard University have constituted themselves an informal committee to solicit contributions to relieve the present financial need of the JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK. The deficit of the JAHRBUCH for the coming fiscal year will amount to about one thousand dollars. The editor, Professor L. Lichtenstein, has appealed for aid. The Emergency Society for German and Austrian Science and Art, which last year appropriated 20,500 marks for the JAHRBUCH, contemplates the continuance of its support, subject to the coöperation of American mathematicians. In view of the importance of the JAHRBUCH, and of the fact that it is virtually indispensable in the pursuit of research, it is hoped that members of the Society will subscribe generously toward its support. Contributions should be sent to Professor W. C. Graustein, 44 Langdon St., Cambridge 38, Mass.

A notable gift for the promotion of mathematical interests in America was announced by the Mathematical Association of America at its summer meeting in September, 1921. Mrs. Mary Hegeler Carus, as trustee for the Edward C. Hegeler Trust Fund, donated to the Association the sum of $1200 annually for five years for the purpose of publishing a series of monographs whose purpose should be to popularize mathematics by making accessible at nominal cost the best thoughts and keenest researches in this field set forth in expository form comprehensible to teachers and students of mathematics and to other readers of mathematical intelligence. The deed of gift includes the promise to capitalize this annual income by a permanent endowment fund if at the end of five years the project shall have proved successful. The trustees of the Association are taking steps to select a board of editors who shall have charge of all details connected with the publication of the proposed monographs.

Among the effects of the late Professor Maxime Bôcher are a considerable number of reprints of his published articles, which his son desires to place in the hands of those who would appreciate them. Members of the Society who desire such reprints should consult the list of Bôcher's writings in this BULLETIN, vol. 25, p. 209 (1919), and send to Professor O. D. Kellogg, Harvard University, Cambridge, Mass., a list by number of the reprints desired.

The Belgian Academy announces the following subjects for prizes in mathematics and physics to be awarded in 1922: A study of the absorption of light in interstellar space; A contribution to the infinitesimal geometry of curved surfaces; A contribution to the theory of the antennæ used in wireless telegraphy. The prize in each case is 1500 francs.

The Richard Lieben prize of the Vienna Academy of Sciences has been awarded to Professor H. Hahn, of the University of Vienna, and Professor J. Radon, of the University of Hamburg.

Professor H. Hahn, of the University of Vienna, has been elected a corresponding member of the Vienna Academy of Sciences.

Professors J. Horn, of the Darmstadt Technical School, A. Kneser, of the University of Breslau, E. Landau, of the University of Göttingen, and E. Salkowski, of the Hannover Technical School, have been elected members of the Leopoldinisch-Carolinische Deutsche Akademie der Naturförscher.

Honorary degrees of doctor of engineering have been conferred on Professor E. Doležal, of the Vienna Technical School, by the Technical School of Aix, and on Professor K. Heun, of the Karlsruhe Technical School, by the Berlin Technical School.

The French Government has conferred the cross of the legion of honor on Professor H. Fehr, dean of the faculty of sciences of the University of Geneva, in recognition of his assistance in enabling French soldiers interned in Switzerland to continue their studies there during the war.

The honorary degree of doctor of science was conferred on Professor Albert Einstein by the University of Manchester.

The University of St. Andrews has conferred the honorary degree of doctor of laws on Professor A. N. Whitehead.

The Royal Society of Edinburgh has elected as honorary fellows Professor Salvatore Pincherle, of the University of Bologna, and Sir Ernest Rutherford.

Sir J. J. Thomson succeeds Sir Richard Glazebrook as president of the Institute of Physics, London.

The University of Edinburgh has conferred the degree of doctor of laws on Dr. Irving Langmuir, of the research laboratory of the General Electric Company, Schenectady. Dr. Langmuir opened the discussion on *The structure of molecules* at the Edinburgh meeting of the British Association.

The University of Groningen has conferred an honorary degree in mathematics and astronomy on Miss Annie J. Cannon, of the Harvard Observatory, in acknowledgment of her work on stellar spectra.

Professor C. N. Haskins has been appointed one of the three representatives of the American Mathematical Society in the Division of Physical Sciences of the National Research Council to succeed Professor L. E. Dickson, whose term expires on July 1, 1922.

Professors Florian Cajori and O. D. Kellogg have been elected fellows of the American Academy of Arts and Sciences.

The following university courses in mathematics are announced:

HARVARD UNIVERSITY (academic year 1921–1922).—All courses meet three times a week throughout the year, except those marked *, which meet for half a year.—By Professor W. F. Osgood: Differential and integral calculus (advanced course); Infinite series and products.*—By Professor C. L. Bouton: The elementary theory of differential equations;* Differential equations, with an introduction to Lie's theory of continuous groups.—By Professor J. L. Coolidge: Vector analysis;* The geometry of the circle.—By Professor O. D. Kellogg: Dynamics (second course).*—By Professor G. D. Birkhoff: The theory of functions (introductory course); The theory of relativity.*—By Professor W. C. Graustein: Differential geometry of curves and surfaces;* Projective geometry;* Introduction to modern geometry.—By Dr. E. Hille: The analytical theory of heat and problems in elastic vibrations.*—By Dr. J. L. Walsh: Introduction to the theory of potential functions and Laplace's equation;* The partial differential equations of mathematical physics.* Professor Kellogg and Dr. Hille will conduct a fortnightly seminar in analysis. Courses of research are also offered by Professor Coolidge in geometry, by Professor Kellogg in the theory of potential functions, by Professor Birkhoff in the theory of differential equations, and by Professor Graustein in geometry.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY (academic year 1921–1922).—By Dr. Norbert Wiener: Fourier's series, two hours.—By Dr. George Rutledge: Theory of functions, two hours.—By Dr. S. D. Zeldin: Vector analysis, two hours.—By Professor H. B. Phillips: Modern electric theory, two hours.—By Professor C. L. E. Moore: Aeronautics, two hours. —By Professor F. S. Woods: Advanced calculus, three hours; Higher geometry, two hours.—By Professor F. L. Hitchcock: Mathematics applied to chemistry, three hours.

RICE INSTITUTE (academic year 1921–1922).—By Dr. H. E. Bray: Projective geometry and linear and quadratic forms.—By Professor P. J. Daniell: Statistical economics; Theory of radiation, electrons, and gravitation.—By Professor G. C. Evans: Integral equations; Theoretical economics, mathematical treatment.—By Professor L. R. Ford: Differential equations.—By Mr. L. E. Ward: Euclidean and non-euclidean differential geometry.

UNIVERSITY OF PENNSYLVANIA (academic year 1921–1922).—By Professor E. S. Crawley: Theory of numbers, three hours (first term).—By Professor G. H. Hallett: Higher algebra, three hours; Galois theory of equations, three hours (second term).—By Professor O. E. Glenn: Invariants, three hours.—By Professor F. H. Safford: Partial differential equations, three hours.—By Professor M. J. Babb: Differential equations, three hours (first term); Functions of a complex variable, three hours (second term).—By Professor G. G. Chambers: Projective geometry, three hours.—By Professor H. H. Mitchell: Probability, three hours (first term); Algebraic numbers, three hours (second term).—By Professor F. W. Beal: Differential geometry, three hours.—By Professor J. R. Kline: Functions of a real variable, three hours (first term); Integrals of Lebesgue and related integrals, three hours (second term).

At the Technical School at Graz, Dr. B. Baule, of the University of Hamburg, has been appointed professor of mathematics, and Dr. E. Kruppa has been promoted to an associate professorship.

Dr. P. Funk has been promoted to an associate professorship of mathematics at the German Technical School at Prague.

At the University of Frankfurt, Professor M. Dehn, of the Breslau Technical School, has been appointed full professor of mathematics. Professor A. Schoenflies has retired from active teaching.

The Hannover Technical School has called Professor R. Grammel, of the Stuttgart Technical School, to a professorship of mathematics. Professor L. Kiepert has retired from active teaching.

Professor H. Grassmann has been promoted to a full professorship of mathematics at the University of Giessen.

Professor O. Haupt, of the University of Rostock, has been appointed to a full professorship at the University of Erlangen.

At the University of Bonn, Professor F. Hausdorff, of the University of Greifswald, has been appointed professor of mathematics. Professor J. O. Müller has been appointed lecturer in algebra, theory of numbers, and theory of probability.

At the University of Hamburg, Professor M. von Laue, of the University of Berlin, has been appointed professor of theoretical physics. Dr. P. Riebesell has been promoted to a professorship.

Professor L. Lichtenstein, of the University of Münster, has been appointed full professor of mathematics at the University of Leipzig.

Professor F. Pfeiffer has been promoted to a full professorship of mathematics at the University of Heidelberg.

Dr. G. Prange has been appointed lecturer in applied mathematics at the University of Halle.

Professor W. Schlink has been appointed professor of technical mechanics at the Darmstadt Technical School.

The following persons have been admitted as privat-docents: Dr. S. Breuer, at the Karlsruhe Technical School; Dr. G. Doetsch, at the Hannover Technical School; Dr. K. Reinhardt, at the University of Frankfurt.

Professor L. Donati, of the University of Bologna, has retired from active teaching.

Professor G. Scorza, of the University of Catania, has been appointed professor of analytic geometry at the University of Naples.

Professors O. Lazzarino and M. Picone, of the University of Cagliari, have been appointed associate professors at the University of Catania. At the University of Cagliari, Dr. Pia Nalli has been appointed associate professor of infinitesimal analysis, and Dr. Lucio Silla associate professor of rational mechanics.

Professor Henri Lebesgue has been appointed professor at the Collège de France, as successor to the late Professor Georges Humbert.

Professor H. C. Plummer, Royal Astronomer of Ireland and professor of astronomy at the University of Dublin, has been appointed professor of mathematics at the Ordnance College, Woolwich.

Mr. H. J. Davis, of University College, Southampton, has been appointed lecturer in mathematics at the Bradford Technical College.

At the University of Alberta, Associate Professor S. D. Killam has been promoted to a full professorship, and Assistant Professor J. W. Campbell to an associate professorship of mathematics.

Dr. H. R. Kingston, formerly of the University of Manitoba, has been appointed professor and head of the department of mathematics at Western University, London, Ontario.

Dr. Theodore Lyman has been appointed to the Hollis professorship of mathematics and natural philosophy at Harvard University, the chair successively held by Benjamin Peirce and Wallace C. Sabine.

At the California Institute of Technology, Professor H. A. Lorentz, of the University of Leyden, will be in residence as lecturer and research associate during two months of the winter of 1921–22. Dr. C. G. Darwin, of Cambridge University, has been appointed professor of mathematical physics for the academic year 1922–23.

Professor G. G. Chambers and H. H. Mitchell have been promoted to full professorships of mathematics at the University of Pennsylvania.

Dean T. F. Holgate, of Northwestern University, has been invited to spend his sabbatical year (1921–22) at the University of Nanking, China, lecturing on mathematical subjects and assisting in the general organization of the University.

At Oberlin College, Dr. C. H. Yeaton, of the Milwaukee College of Engineering, has been appointed assistant professor of mathematics. Mr. F. E. Carr has been promoted to an assistant professorship.

Mr. L. S. Hill has been appointed associate professor of mathematics at the University of Maine.

Dr. W. E. Edington, of the University of Illinois, has been appointed assistant professor of mathematics at Pennsylvania State College.

Professor M. O. Tripp, of the University of Maine, has been appointed professor of mathematics at Wittenberg College.

Dr. D. V. Steed, of the University of California, has been appointed assistant professor of mathematics at the University of Southern California.

Mr. E. L. Mackie, of Harvard University, has been appointed assistant professor of mathematics at the University of North Carolina.

Dr. C. G. P. Kuschke has been appointed assistant professor of mathematics and physics at the University of Porto Rico.

Professor Eva Chandler, of Wellesley College, has retired from active teaching.

Dr. C. N. Reynolds, of Dartmouth College, has been appointed assistant professor of mathematics at the University of West Virginia.

Professor Gertrude I. McCain, of Oxford College, Oxford, Ohio, has been appointed professor of mathematics at Westminster College, New Wilmington, Pa.

President R. J. Aley, of the University of Maine, formerly professor of mathematics and dean at the University of Indiana, has been appointed president of Butler College, Indianapolis.

Professor J. D. Bond, of the Agricultural and Mechanical College of Texas, has been appointed associate professor of mathematics at Louisiana State University.

Dr. F. R. Morris, of the University of California, has been made head of the department of mathematics at the State Teachers and Junior College at Fresno.

Professor G. H. Cresse, of the U. S. Naval Academy, has been appointed associate professor of mathematics at the University of Arizona.

Mr. E. T. Browne has been appointed assistant professor of mathematics at Trinity College.

Professor T. O. Walton, of Michigan Agricultural College, has been appointed assistant professor of mathematics at Kalamazoo College.

Dr. E. S. Hammond, of Princeton University, has been appointed assistant professor of mathematics at Bowdoin College.

At Purdue University, Professor William Marshall has been made acting head of the department of mathematics; Mr. F. H. Hodge and Mr. C. S. Doan have been appointed assistant professors.

At Kansas State Agricultural College, Miss Bess J. McKittrick, Miss Emma Hyde and Mr. C. S. Lewis have been promoted to assistant professorships.

At Northwestern University, Associate Professor E. J. Moulton has been promoted to a full professorship.

At Johns Hopkins University, Dr. F. D. Murnaghan has been promoted to an associate professorship of applied mathematics.

At Yale University, Professor W. A. Wilson has been promoted to a full professorship.

At Lehigh University, Assistant Professors J. E. Stocker and J. B. Reynolds have been promoted to associate professorships of mathematics and astronomy.

At Smith College, Assistant Professor Susan R. Benedict has been promoted to a full professorship.

At Dickinson College, Professor W. W. Landis has leave of absence, which he is spending in Europe. Professor N. B. Rosenberger is acting head of the Department of Mathematics.

At the University of Minnesota, Associate Professor W. F. Holman has been promoted to a full professorship, and Assistant Professor H. H. Dalaker has been promoted to an associate professorship.

At Kalamazoo College, Professor T. O. Walton of the Michigan Agricultural College has been appointed assistant professor.

At the University of Wisconsin, Assistant Professor A. Dresden has been promoted to an associate professorship.

At Colorado College, Associate Professor W. V. Lovitt has been promoted to a full professorship.

At Iowa State College, Professor Maria M. Roberts has been appointed Dean of the Junior College; Assistant Professor Helen Tappan has been promoted to an associate professorship; Professor E. R. Smith of Pennsylvania State College has been appointed head of the Department; Dr. E. S. Allen of West Virginia University has been appointed associate professor; Dr. J. S. Turner has been appointed assistant professor. Professor J. V. McKelvey, whose appointment at Westminster College was previously announced (this Bulletin, vol. 27, p. 337), will remain at Iowa State College.

Mr. E. H. Vance has been appointed professor of mathematics at Defiance College.

The following recent appointments to instructorships in mathematics at American colleges and universities are announced:

Bradley Polytechnic Institute, Mr. A. E. Gant;
Bucknell University, Mrs. Theron Clark;
University of California, Dr. P. H. Daus;
Colgate University, Mr. H. A. Dobell;
Colorado College, Miss W. M. Spingler;
Harvard University, Mr. H. W. Brinkman, Mr. R. M. Foster, Dr. C. E. Hille (Peirce instructor), Mr. Lincoln LaPaz, Dr. J. L. Walsh (permanent instructor);
Hood College, Miss Rachel T. Easterbrooks;
University of Illinois, Dr. C. C. Camp, Dr. J. M. Stetson, Dr. Beulah Armstrong;
Iowa State College, Mr. A. L. Young;
Knox College, Miss Helen Calkins;
Mills College, Dr. Nina M. Alderton;
University of Minnesota, Mr. C. G. Phipps;
University of Missouri, Miss Katherine Wyant;
University of Pennsylvania, Mr. J. D. Eshleman;
Princeton University, Dr. C. C. MacDuffee, Dr. E. L. Post;
Purdue University, Mr. J. W. Branson, Mr. E. G. Keller, Mr. J. Knox, Mr. J. H. Shock;
Rice Institute, Mr. L. E. Ward;
University of Rochester, Mr. D. E. Whitford;
Rose Polytechnic Institute, Mr. A. J. Bedard;
University of Texas, Mr. R. L. Wilder;
Union College, Mr. Ralph Bennett, Mr. A. D. Snyder;
Washington and Jefferson College, Mr. C. J. Cardin;

Wellesley College, Miss Rachel Blodgett, Miss Ruby Willis;
Western College for Women, Miss May B. Carter;
University of Wisconsin, Mr. M. L. MacQueen, Mr. E. B. Miller, Mr. A. R. Wapple;
Yale University, Dr. M. C. Foster.

Dr. A. W. A. Thaer, director of the Oberrealschule at Hamburg, died March 1, 1921, at the age of sixty-six years.

Professor J. K. Thomae, of the University of Jena, died April 1, 1921, at the age of eighty-one years.

The death is reported of Professor V. Dantschler von Kollesburg, of the University of Graz, at the age of seventy-three years.

The death is reported of Professor N. Joukowski, of the Moscow Technical School.

The death is reported of Professor A. Riggenbach-Burckhardt, of the University of Basel.

J. A. Gaillot, formerly sub-director of the Paris Observatory and collaborator with Leverrier in the preparation of his tables, died at Chartres, June 4, 1921, at the age of eighty-seven years.

Professor Gabriel Lippman, of the department of physics of the University of Paris, died on June 13, 1921, at the age of seventy-five years.

Mr. H. T. Gerrans, of Oxford University, died June 20, 1921. Mr. Gerrans had been a member of the American Mathematical Society since 1901.

Professor Hermann Amandus Schwarz of Berlin died November 30, 1921. Born January 25, 1843, he studied at Berlin, receiving his doctorate in 1864. He taught at Halle, Zürich, Göttingen, and Berlin. He held the honorary doctorate of the University of Christiania and was a member of the learned societies of Berlin, Göttingen, and Paris. He published a number of memoirs on subjects of analysis and differential geometry. His collected works appeared in 1890.

Professor Camille Jordan of the Collège de France and the École Polytechnique, died January 21, 1922. He was born at Lyon, January 5, 1838. He received the doctorate at Paris in 1861. He was widely known for his important and valuable contributions to analysis, algebra and geometry. He was a member of the Institute and of a large number of mathematical societies, and held honorary degrees from Christiania and Louvain. He was editor of the Journal de Mathématiques Pures et Appliquées. He served as Vice-President of the Fourth International Congress of Mathematicians at Rome, 1908.

Professor Nathaniel French Davis, Emeritus, of Brown University, died May 17th, 1921. Professor Davis was born at Laconia, New Hampshire, June 11th, 1847. He attended Brown University and the University of Göttingen, and held the degree of LL.D. from Colby College. After teaching in secondary and normal schools, he came to Brown University as instructor in 1874 and became assistant professor in 1879, associate professor in 1889, professor in 1890, and emeritus professor in 1915. Professor Davis had been a member of the American Mathematical Society since 1891.

Professor Alfred Monroe Kenyon, head of the Department of Mathematics at Purdue University, died on July 27th, 1921. Professor Kenyon was born at Medina, Ohio, December 10th, 1869. He attended Hiram College, Case School of Applied Science, and Harvard University. After teaching in a secondary school, at Case School of Applied Science, and at the Massachusetts Nautical Training School, he came to Purdue as instructor in 1898, became associate professor in 1901, professor in 1903, and head of the Department in 1908. He was the author of several college text-books. Professor Kenyon had been a member of the American Mathematical Society since 1900.

Professor Henry Turner Eddy, Emeritus, of the University of Minnesota, died December 18, 1921. He was born at Stoughton, Mass., June 9, 1844. He studied at Yale, Cornell, Berlin and Paris, receiving his doctorate at Cornell in 1872. He held honorary degrees from Central College and Yale. He taught at Yale, Cornell, and Princeton, was Dean and later President at Cincinnati, and served as President of Rose Polytechnic Institute from 1890 to 1894. He came to Minnesota as Professor in 1894, and became Dean of the Graduate School in 1906. He retired from both positions with the title Emeritus in 1912. Professor Eddy was Secretary of Section A of the American Association for the Advancement of Science in 1882, and Vice-President of the Association in 1884. He was President of the Society for the Promotion of Engineering Education in 1896. He had been a member of the American Mathematical Society since 1891.

Professor Wooster Woodruff Beman of the University of Michigan died January 18, 1921. He was born May 28, 1850. He studied at the University of Michigan, receiving his master's degree in 1873. He taught at Kalamazoo College and at Michigan, where he became full Professor in 1887. He had been a member of the American Mathematical Society since 1891. He was Secretary of Section A of the American Association for the Advancement of Science in 1890, and Vice-President in 1897. He was well known as joint author of a number of elementary text-books and as a translator of several famous monographs.

Professor Charles Leonard Bouton of Harvard University died February 20, 1922. He was born at St. Louis, April 25, 1869. After receiving his mathematical training at Washington University, Harvard and Leipsic, he taught at Washington University and at Harvard. He had been a member of the American Mathematical Society since 1898, and served on the editorial staff of the BULLETIN and the TRANSACTIONS.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

Bachmann (P.). Niedere Zahlentheorie. 1ter Teil. Anastatischer Neudruck. Leipzig, 1921.

Baumgartner (L.). Gruppentheorie. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921.

Bianchi (L.). Lezioni sulla teoria dei numeri algebrici e principî d'aritmetica analitica. Corso d'analisi 1920–1921. Pisa, 1921. 8vo. 8 + 444 pp.

Carslaw (H. S.). Introduction to the theory of Fourier's series and integrals. 2d edition, completely revised. London, Macmillan, 1921. 8vo. 12 + 323 pp. 30s.

Corral (J. I.). Nuevos teoremas que resuelven el problema de Hurwitz. Madrid, Imprenta Clásica Española, 1921.

Encyklopädie der mathematischen Wissenschaften. Band II 2, Heft 5: E. Hilb, Nichtlineare Differentialgleichungen. A. Krazer and W. Wirtinger, Abelsche Funktionen und allgemeine Thetafunktionen. Titel, Inhaltsverzeichnis und Register zu Band II 2. Leipzig, Teubner, 1921.

Feyer (E.). Asymptotische Darstellung gewisser meromorpher Funktionen. (Diss.) Breslau, 1919.

Halphen (G. H.). Œuvres. Tome 3. Paris, Gauthier-Villars, 1921. 8vo. 12 + 518 pp. Fr. 90.00

Hilb (E.). See Encyklopädie.

Isenkrahe (C.). Untersuchungen über das Endliche und das Unendliche mit Ausblicken auf die philosophische Apologetik. Bonn, Marcus und Webers Verlag, 1920. In 3 Bänden. 8 + 224 + 8 + 230 + 12 + 247 pp.

Krazer (A.). See Encyklopädie.

Lecat (M.). Bibliographie des séries trigonométriques avec un appendice sur le calcul des variations. Bruxelles, chez l'auteur, 1921. 8vo. 8 + 168 pp. Fr. 25.00

Löffler (E.). See Severi (F.).

Pairman (E.). Tables of the digamma and trigamma functions. (Tracts for Computers, edited by Karl Pearson, No. 1.) Cambridge, University Press, 1919. 8vo. 19 pp. 3s.

Pearson (K.). See Pairman (E.).

Perron (O.). Irrationalzahlen. Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 8 + 186 pp.

Pringsheim (A.). Vorlesungen über Zahlen- und Funktionenlehre. Band 1: Zahlenlehre. 3te Abteilung: Komplexe Zahlen. Reihen mit komplexen Gliedern. Unendliche Produkte und Kettenbrüche. Leipzig, Teubner, 1921.

Runge (C.). Praxis der Gleichungen. 2te, verbesserte Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 2 + 172 pp.

Scheffers (G.). Einführung in die Theorie der Kurven in der Ebene und im Raume. 2te, verbesserte und vermehrte Auflage. Anastatischer Neudruck. Berlin, 1921.

SEVERI (F.). Vorlesungen über algebraische Geometrie. Geometrie auf einer Kurve. Riemannsche Flächen. Abelsche Integrale. Berechtigte Deutsche Uebersetzung von Dr. Eugen Löffler. Leipzig, Teubner, 1921. 8vo. 16 + 408 pp.

WEBER (H.). Lehrbuch der Algebra. Kleine Ausgabe in einem Bande. 2ter unveränderter Abdruck. Braunschweig, Vieweg, 1921. 10 + 528 pp.

WIRTINGER (W.). See ENCYKLOPÄDIE.

II. ELEMENTARY MATHEMATICS

BALL (W. W. R.). String figures. 2d edition. Cambridge, University Press, 1921. 69 pp. 2s. 6d.

BALSER (L.). See NELL (A. M.).

BARDEY (E.). See LIETZMANN (W.).

BERTIN (C.). La droite radiogoniométrique. Paris, Gauthier-Villars, 1921. 8vo. 12 pp. Fr. 2.00

DRESDEN (A.). Plane trigonometry. New York, Wiley, 1921. 8 + 110 pp.

EDWARDS (J.). A treatise on the integral calculus, with applications, examples, and problems. Volume 1. London, Macmillan, 1921. 21 + 907 pp. 50s.

EMCH (H.). Mathematik in der Natur. Zürich, Rascher, 1921. 16mo. 86 pp.

ERNST (E.). Mathematische Unterhaltungen und Spielereien für Schüler und Freunde der Mathematik. Band 2. Ravensberg, 1921.

GRAUSTEIN (W. C.). See OSGOOD (W. F.).

HAYASHI (K.). Fünfstellige Tafeln der Kreis- und Hyperbelfunktionen sowie der Funktionen e^x und e^{-x} mit den natürlichen Zahlen als Argument. Berlin, Vereinigung wissenschaftlicher Verleger, 1921.

HOKE (K. J.). See WILSON (G. M.).

JUNG (J.). Der Ueberschlag. Ein rasches Näherungsverfahren zum Berechnen von Summen, besonders bei grosser Postenzahl. Wien, Gerold, 1919.

LIETZMANN (W.). Bardey-Lietzmann, Aufgabensammlung für Arithmetik, Algebra und Analysis. Reformausgabe B für Realanstalten, Unterstufe. 4te Auflage. Leipzig, Teubner, 1920. 220 pp.

NELL (A. M.). Fünfstellige Logarithmen der Zahlen und der trigonometrischen Funktionen. Vollige Neubearbeitung von L. Balser. Giessen, 1920.

D'OCAGNE (M.). Instruction sur l'usage de la règle à calcul. 2e édition. Paris, Gauthier-Villars, 1921. 8vo. 8 pp. Fr. 1.50

OSGOOD (W. F.) and GRAUSTEIN (W. C.). Plane and solid analytic geometry. New York, Macmillan, 1921. 18 + 614 pp.

SCOTT (P. W.). Elements of practical geometry. A two years' course for day and technical evening students. London, Pitman, 1921. 5 + 185 pp. 5s.

WERKMEISTER (P.). Praktisches Zahlenrechnen. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921.

WIJDENES (P.). Lagere algebra. Deel I: De algebraische grootheden en hare bewerkingen. Antwoorden en uitwerkingen van de vraagstukken uit Lagere algebra, I. Groningen, Noordhoff, 1919–1920. 8vo. 287 + 60 pp. Fl. 4.68 + 2.00

WILSON (G. M.) and HOKE (K. J.) How to measure. London and New York, Macmillan, 1921. 7 + 285 pp. 12s.

III. APPLIED MATHEMATICS

ADAMS (O. S.). Latitude developments connected with geodesy and cartography with tables including a table for Lambert equal-area meridional projection. (U. S. Coast and Geodetic Survey, Special Publication No. 67.) Washington, Government Printing Office, 1921. 132 pp.

D'ALEMBERT (J.). Traité de dynamique. 2 volumes. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1921. 16mo. 40 + 102 pp. Fr. 6.00

BAUER (G.). Die Helmholtzsche Wirbeltheorie für Ingenieure bearbeitet. München, 1919.

BAUSCHINGER (J.). See ENCYKLOPÄDIE.

BEER, F. Die Einsteinsche Relativitatstheorie und ihre historisches Fundament. Sechs Vortrage fur Laien. 4te Auflage. Wiem. 1920.

BLOCH (E.). Théorie cinétique des gaz. Paris, Colin, 1921. 16mo. 176 pp. Fr. 5.00

BRODETSKY (S.). The mechanical principles of the aeroplane. London, J. and A. Churchill, 1921. 7 + 272 pp. 21s.

CHARBONNIER (P.). Traité de balistique extérieure. Tome 1: Les théorèmes généraux de la balistique. Paris, G. Doin et Gauthier-Villars, 1921. 8vo. 9 + 637 pp. Fr. 75.00

CHATLEY (H.). A text-book of Aeronautical Engineering. The problem of flight. 3d edition. London, Griffin, 1921.

CRATHORNE (A. R.). See RIETZ (H. L.).

CZUBER (E.). Die statistischen Forschungsmethoden. Wien, Seidel, 1921. 8vo. 250 pp.

DENT (J. A.) and HARPER (A. C.). Kinematics and kinetics of machinery. New York, Wiley, 1921.

EDDINGTON (A. S.). Espace, temps, gravitation. Traduit de l'anglais par J. Rossignol. Introduction de P. Langevin. Paris, Hermann, 1921. 8vo. 430 pp. Fr. 28.00

EINSTEIN (A.). Geometrie und Erfahrung. Berlin, Springer, 1921.

——. See FREUNDLICH (E.).

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band VI 2, Heft 7: K. Laves, Die Satelliten. J. Bauschinger, Bestimmung und Zusammenhang der astronomischen Konstanten. Leipzig, Teubner, 1920.

FREUNDLICH (E.). Die Grundlagen der Einsteinschen Gravitationstheorie. Mit einem Vorwort von A. Einstein. 4te, erweiterte und verbesserte Auflage. Berlin, Springer, 1920.

FUNK (P.). Die linearen Differenzengleichungen und ihre Anwendungen in der Theorie der Baukonstruktionen. Berlin, Springer, 1920. 84 pp.

GOUARD (E.) et HIERNAUX (G.). Mécanique, précédée d'une étude sur les graphiques. Paris, Dunod, 1920. 8vo. 148 pp. Fr. 11.00

GRAMMEL (R.). Der Kreisel. Seine Theorie und seine Anwendungen. Braunschweig, Vieweg, 1920. 10 + 350 pp.

GRIBBLE (T. G.). Handy methods of geodetic astronomy for land surveyors. London, Potter, 1921. 28 pp.

GRIMSEHL (E.). Lehrbuch der Physik. Band 1: Mechanik, Akustik, Wärmelehre und Optik. 4te Auflage. Leipzig, Teubner, 1920. 1011 pp.

HAMEL (G.). Mechanik. I: Grundbegriffe der Mechanik. (Aus Natur und Geisteswelt, Band 684.) Leipzig, Teubner, 1921. 132 pp.

HARPER (A. C.). See DENT (J. A.).

HARROW (B.). From Newton to Einstein. Changing conceptions of the universe. London, Constable, 1920. 10 + 95 pp. 2s. 6d.

HEATH (T. L.). The Copernicus of antiquity (Aristarchus of Samos). (Pioneers of Progress: Men of Science Series.) London, Society for Promoting Christian Knowledge, and New York, Macmillan, 1921. 12mo. 60 pp. 2s. 6d.

HIERNAUX (G.). See GOUARD (E.).

JAEGER (F. M.). Lectures on the principle of symmetry and its applications in all natural sciences. 2d edition. Amsterdam, "Elsevier," 1920. 12 + 348 pp.

JANET (P.). Leçons d'électrotechnique générale. Paris, Gauthier-Villars. Tome 1: Généralités. Courants continus. 5e édition. 1921. 8 + 452 pp. Tome 3: Moteurs à courants alternatifs. Couplage et compoundage des alternateurs. Transformateurs polymorphiques. 4e édition. 1920. 4 + 388 pp. Fr. 45.00 + 36.00

JEANS (J. H.). The dynamical theory of gases. 3d edition. Cambridge, University Press, 1920. 7 + 442 pp. 30s.

KAHL (H.). Das luftelektrische Potentialgefälle als Funktion der Höhe. (Diss., Kiel.) Kiel, Druck von H. Fiencke, 1915. 8vo. 56 pp.

KEAL (H. M.) and LEONARD (C. J.). Mathematics for electrical students. New York, Wiley, 1921.

KIRCHBERGER (P.). Mathematische Streifzüge durch die Geschichte der Astronomie. (Mathematisch-physikalische Bibliothek, Band 40.) Leipzig, Teubner, 1921. 4 + 54 pp.

——. Was kann man ohne Mathematik von der Relativitätstheorie verstehen? Mit einem Geleitwort von M. von Laue. 2te, vermehrte Auflage. Karlsruhe, 1921.

KOPFF (A.). Grundzüge der Einsteinschen Relativitätstheorie. Leipzig, Hirzel, 1921. 8 + 198 pp.

LADENBERG (R.). Plancks elementares Wirkungsquantum und die Methoden zu seiner Bestimmung. Leipzig, 1921.

LAMB (H.). Higher mechanics. Cambridge, University Press, 1920. 10 + 272 pp. 25s.

LÄMMEL (R.). Die Grundlagen der Relativitätstheorie. Populärwissenschaftlich dargestellt. Berlin, Springer, 1921. M. 14.00

LANGEVIN (P.). See EDDINGTON (A. S.).

von Laue (M.). Die Relativitätstheorie. Band 1: Das Relativitätsprinzip der Lorenztransformation. 4te vermehrte Auflage. Braunschweig, Vieweg, 1921. 13 + 302 pp.

——. See Kirchberger (P.).

Laves (K.). See Encyklopädie.

Leonard (C. J.). See Keal (H. M.).

Love (A. E. H.). Theoretical mechanics. An introductory treatise on the principles of dynamics. 3d edition. Cambridge, University Press, 1921. 15 + 310 pp. 30s.

Ludendorff (H.). See Wislicenus (W. F.).

MacDougall (F. H.). Thermodynamics and chemistry. New York, Wiley, and London, Chapman and Hall, 1921. 5 + 391 pp. 30s.

Mach (E.). Die Prinzipien der physikalischen Optik. Leipzig, Barth, 1921. 10 + 444 pp.

Maggi (G. A.). Dinamica fisica. Lezioni sulle leggi generali del movimento dei corpi naturali. 3a edizione. Pisa, Spoerri, 1921.

——. Dinamica dei sistemi. Pisa, 1921. 8vo. 7 + 408 pp.

Masius (M.). See Rougier (L.).

Michaud (F.). Energétique générale. Paris, Gauthier-Villars, 1921. 8vo. 7 + 229 pp. Fr. 20.00

Milankovitch (M.). Théorie mathématique des phénomènes thermiques produits par la radiation solaire. Paris, Gauthier-Villars, 1920. 8vo. 16 + 340 pp. Fr. 20.00

Newbigin (M. I.). Ordnance survey maps; their meaning and use. 2d edition. London, Macmillan, 1920.

Nilsson (M. P.). Primitive time-reckoning. A study in the origins and first development of the art of counting time among the primitive and early culture people. Lund, Gleerup, and London, Oxford University Press, 1920. 13 + 384 pp. 21s.

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Oppenheim (S.). Das astronomische Weltbild im Wandel der Zeit. II: Moderne Astronomen. 2te Auflage. (Aus Natur und Geisteswelt, Nr. 445.) Leipzig, Teubner, 1920.

Patschke (A.). Umstutz der Einsteinschen Relativitätstheorie. Einführung in die einheitliche Erklärung und Mechanik der Naturkräfte . Berlin-Wilmersdorf, Selbstverlag, 1920.

Paynter (J. E.). Practical geometry for builders and architects. London, Chapman and Hall, 1921. 12 + 409 pp. 15s.

Petrovitch (M.). Mécanismes communs aux phénomènes disparates. Paris, Alcan, 1921. 280 pp. Fr. 8.00

Petzold (J.). Die Stellung der Relativitätstheorie in der geistigen Entwicklung der Menschheit. Dresden, Sybillen-Verlag, 1921.

Planck (M.). Die Enstehung und bisherige Entwicklung der Quantentheorie. Leipzig, Barth, 1920. 8vo. 32 pp.

Rietz (H. L.), Crathorne (A. R.) and Rietz (J. C.). Mathematics of finance. New York, Holt, 1921. 14 + 280 pp.

Rietz (J. C.). See Rietz (H. L.).

ROBB (A. A.). The absolute relations of time and space. Cambridge, University Press, 1921. 9 + 80 pp. 5s.

ROSSIGNOL (J.). See EDDINGTON (A. S.).

ROUGIER (L.). Philosophy and the new physics. An essay on the relativity theory and the theory of quanta. Authorized translation from the author's corrected text of "La matérialisation de l'énergie" by M. Masius. Philadelphia, Blakiston, 1921. 16mo. 16 + 159 pp. $1.75

SCHLESINGER (L.). Raum, Zeit und Relativitätstheorie. Gemeinverständliche Vorträge. Leipzig, Teubner, 1920. 40 pp.

SCHLÜTER (W.). Beitrag zur Geschichte der Basisvergrösserung und zur Theorie der zweckmässigsten Form des Vergrösserungsdreieckes. (Diss., Karlsruhe.) Mittweida, Selbstverlag, 1920.

SCHMID (T.). Darstellende Geometrie. 2ter Band. (Sammlung Schubert, Nr. 66.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 315 pp. M. 33.00

SCHNEIDER (I.). Das Raum-Zeit-Problem bei Kant und Einstein. Berlin, Springer, 1921. M. 12.00

SCHWARZSCHILD (K.). Ueber den Einfluss von Wind und Luftdichte auf die Flugbahn der Geschosse. Berlin, 1920.

SEYRIG (T.). Statique graphique des systèmes triangulés. 3e édition. Tome 1: Exposés théoriques. Tome 2: Exemples d'applications. Paris, Gauthier-Villars, 1918. 8vo. Fr. 3.75 + 3.75

SLOSSON (E. E.). Easy lessons in Einstein. A discussion of the non-intelligible features of the theory of relativity. London, Routledge, and New York, Harcourt, Brace and Howe, 1920. 7 + 128 pp. 5s.

TAYLOR-JONES (E.). The theory of the induction coil. London, Pitman, 1921. 11 + 217 pp. 12s. 6d.

THOMSON (J. J.). Elements of the mathematical theory of electricity and magnetism. 5th edition. Cambridge, University Press, 1921. 8 + 410 pp. 30s.

UBACH (J.). La teoria de la relatividad en la fisica moderna. Buenos Aires, S. de Amorrortu, 1920. 8vo. 45 pp.

VALENTINER (S.). Anwendungen der Quantenhypothese in der kinetischen Theorie der festen Körper und der Gase. In elementarer Darstellung. 2te erweiterte Auflage. Braunschweig, Vieweg, 1921. 6 + 90 pp.

WANGERIN (A.). Theorie des Potentials und der Kugelfunktionen. 2ter Band. (Sammlung Schubert, Nr. 59.) Berlin, Vereinigung wissenschaftlicher Verleger 1921. 8 + 286 pp. M. 60.00

WEATHERBURN (C. E.). Elementary vector analysis: with application to geometry and physics. London, Bell, 1921. 27 + 184 pp. 12s.

WEYL (H.). Raum, Zeit, Materie. Vorlesungen über allgemeine Relativitätstheorie. 4te, erweiterte Auflage. Berlin, Springer, 1921. M. 48.00

WIENER (O.). Fliegerkraftlehre. Leipzig, Hirzel, 1920. 240 pp.

WISLICENUS (W. F.) und LUDENDORFF (H.). Astrophysik. (Sammlung Göschen, Nr. 91.) Berlin, Vereinigung wissenschaftlicher Verleger, 1920. M. 4.20

MATHEMATICS
DEPARTMENT

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The two hundred nineteenth regular meeting of the Society was held at Columbia University on Saturday, October 29, 1921, extending through the usual morning and afternoon sessions. The attendance included the following forty members:

Archibald, Babb, Blake, Bowden, Cole, G. M. Conwell, Eisenhart, Fite, Philip Franklin, Gill, Glenn, Gronwall, Grove, Haskins, Hebbert, Joffe, Kasner, Kellogg, Kircher, Kline, Lamson, McDonnell, MacDuffee, MacNeish, H. H. Mitchell, Northcott, Oglesby, Pfeiffer, Post, Reddick, R. G. D. Richardson, Ritt, Seely, Siceloff, P. F. Smith, Veblen, H. S. White, Whittemore, E. C. Williams, J. W. Young.

At the meeting of the Council, the following thirty persons were elected to membership in the Society:

Mr. Clarence Raymond Adams, Harvard University;
Professor Edward Tankard Browne, Trinity College;
Professor Frank William Bubb, Washington University;
Mr. Curtis Corsair Carter, Bluffs, Scott County, Ill.;
Mr. Joel David Eshleman, University of Pennsylvania;
Mr. Matthew Moses Feldstein-Tartakovsky, Quartermaster Corps, U. S. Army;
Mr. Harold Joseph Gay, Worcester Polytechnic Institute;
Professor Lachlan Gilchrist, University of Toronto;
Professor Edward Sanford Hammond, Bowdoin College;
Mr. Edward H. Hezlett, Connecticut General Life Insurance Company;
Professor Joseph Ellis Hodgson, West Virginia University;
Miss Nelle Louise Ingels, Bureau of Statistics, Interstate Commerce Commission;
Mr. John Melvin Laird, Connecticut General Life Insurance Company;
Mr. Rudolph Ernest Langer, Harvard University;
Professor Michael Alexander Mackenzie, University of Toronto;
Professor Edmund Sewall Manson, Ohio State University;
Mr. Louis Edward Mensenkamp, High School, Freeport, Ill.;
Dr. Raymond Kurtz Morley, Worcester Polytechnic Institute;
Miss Eleanor Pairman, Broomieknowe, Scotland;
Miss Echo Dolores Pepper, University of Washington;
Mr. Charles Scott Porter, Worcester Polytechnic Institute;
Professor William Walter Rankin, Agnes Scott College;

Mr. Noah Bryan Rosenberger, Philadelphia, Pa.;
Mr. Norris Edward Sheppard, University of Toronto;
Professor George Eulas Foster Sherwood, University of California, Southern Branch;
Miss May Julia Sperry, Knox College;
Professor John Lighton Synge, University of Toronto;
Professor John Sidney Turner, Alabama Polytechnic Institute;
Mr. Correa Moylan Walsh, Bellport, L. I.;
Professor Thomas Orr Walton, Kalamazoo College.

One application for membership was received.

A committee consisting of Mr. S. A. Joffe and Professor P. H. Linehan was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. It was voted to publish a List of Officers and Members for 1921–1922.

It was voted to accept the invitation of Harvard University to hold the annual meeting of 1922 at Cambridge.

Vice-president Cole presided at the morning session of the Society, relieved by Professor P. F. Smith in the afternoon. Titles and abstracts of the papers read at this meeting follow below. The papers of Mr. Rice, Dr. Walsh, Mr. Robinson, and Dr. Campbell were read by title. Professor R. L. Moore's papers were read by Professor Kline.

1. Professor J. K. Whittemore: *Total geodesic curvature.*

The author shows that the rate of turning, per unit length of arc of a curve on a surface, of the plane determined by the tangent to the curve and the normal to the surface (called by Professor Osgood* the bending of the curve) is given by

$$\sqrt{1/\rho_g^2 + 1/\tau_g^2},$$

where ρ_g and τ_g are the radii of geodesic curvature and geodesic torsion respectively.

2. Professor J. F. Ritt: *On the composition of polynomials.*

A polynomial $F(z)$ of degree greater than unity is here called *composite* if there exist two polynomials $\varphi_1(z)$ and $\varphi_2(z)$, each of degree greater than unity, such that $F(z) = \varphi_1[\varphi_2(z)]$.

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* This Bulletin, vol. 27, p. 391.

Otherwise $F(z)$ is called *prime.* Consider a decomposition of $F(z)$ into prime polynomials,

$$F = \varphi_1\varphi_2 \cdots \varphi_r,$$

in which each $\varphi_i(z)$ is understood to be substituted for z in the polynomial which precedes it. Two such decompositions are considered equivalent if the second can be obtained from the first by inserting suitable linear polynomials between the prime polynomials. In this paper the circumstances are completely determined under which a polynomial can have two distinct decompositions into prime polynomials. If there are two such distinct decompositions, they contain the same number of polynomials, and the degrees of the polynomials in one decomposition are the same as those in the other, except for the order in which they occur. The full statement of results is simple, but somewhat too long for an abstract.

3. Professor J. F. Ritt: *Complete determination of polynomials whose inverses can be expressed in terms of radicals.*

In an earlier paper the writer determined the structure of those Riemann surfaces with a prime number of sheets which support functions expressible in terms of radicals. For equations of composite degree, this paper determines all polynomials whose inverses can be expressed in terms of radicals. If a polynomial $F(z)$, with inverse so expressible, is decomposed into prime polynomials, in the manner explained in the preceding abstract, then each prime polynomial is either of the fourth degree, or else is of the form $\lambda_1\{\pi[\lambda_2(z)]\}$, where $\lambda_1(z)$ and $\lambda_2(z)$ are linear, and where $\pi(z)$ is either a prime power of z, or else a trigonometric polynomial of prime degree.*

4. Professor R. L. Moore: *Concerning continuous curves in the plane.*

In this paper the following theorems are established: (I) If N is a closed proper subset of a continuous curve M and two points of $M - N$ can be joined by a connected subset of $M - N$ then they can be joined by a simple continuous arc which is a subset of $M - N$; (II) If, in a plane, two points are separated from each other by a continuous curve M then they are separated from each other by a simple closed curve which is a subset of M. The first of these theorems is an extension of a theorem proved in a former paper (this Bulletin, vol. 23 (1917), pp. 233–236).

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* $\pi(z)$ is a trigonometric polynomial of degree n if $\cos nu = \pi(\cos u)$.

5. Professor R. L. Moore: *Concerning the relation of a continuous curve to its complement in space of three dimensions.*

According to a theorem of Schoenflies, in order that a point set M lying in a plane S should be a continuous curve it is necessary and sufficient that (1) every point of the boundary of any one of the domains complementary to M should be reachable "from all sides" with respect to that domain, (2) if $\epsilon > 0$ there do not exist in $S - M$ infinitely many mutually exclusive domains $R_1, R_2, R_3, \cdots$ all of diameter more than ϵ and such that each R_n has a portion of M as its boundary. In the present paper an example is given of a continuous curve in space of three dimensions which satisfies neither of these conditions. It is also shown that a similar criterion of Jordan and Schoenflies that a plane point set of more special type be a continuous curve is in the case of three dimensions neither necessary nor sufficient. It is true, however, that a closed, bounded and connected point set in a space of three dimensions is a continuous curve provided it is the common boundary of two mutually exclusive domains both of which are uniformly connected im kleinen. It is not sufficient to assume merely that one of these domains has this property.

6. Professor Edward Kasner: *An algebraic solution of Einstein's cosmological equations.*

Professor Kasner discusses the gravitational equations in Einstein's latest cosmological form $G_{ik} - \frac{1}{4}g_{ik}G = 0$. If we require the quaternary form ds^2 to be the sum of two binary forms, that is, the sum of the squared elements of two surfaces, then the only cosmological solution (neglecting the trivial euclidean form) is found to be

$$ds^2 = x_1^{-2}(dx_1^2 + dx_2^2) + x_3^{-2}(dx_3^2 + dx_4^2).$$

This represents a quartic manifold of four dimensions embedded in a 6-flat. The finite equations, in six cartesian coördinates, are

$$X_1^2 + X_2^2 + X_3^2 = 1, \qquad X_4^2 + X_5^2 + X_6^2 = 1.$$

This is probably the simplest solution of Einstein's equations which has thus far been found, and the first one (beyond the obvious flat and spherical spaces) which in its finite form is algebraic. In an earlier paper solutions have been found where the ten g_{ik} are algebraic functions, but these do not usually represent algebraic spreads. (See SCIENCE, Sept. 30, 1921, pp. 304–305, and a forthcoming paper in the MATHEMATISCHE ANNALEN.)

7. Dr. T. H. Gronwall: *On biharmonic functions.*

A biharmonic function satisfies the equation $\Delta\Delta u = 0$ or $\partial^4 u/\partial x^4 + 2\partial^4 u/\partial x^2 \partial y^2 + \partial^4 u/\partial y^4 = 0$ and is regular inside a curve C, the values of u and $\partial u/\partial n$ on C being given. Dr. Gronwall proves that when $\partial u/\partial n \geqq 0$ everywhere on C, then u has at least one maximum but no minimum inside C. A number of other qualitative results are also proved, most of which have immediate physical applications. Some of these results were conjectured, but not proved, by Hadamard in his prize memoir on elastic plates (MÉMOIRES DES SAVANTS ÉTRANGERS, 1909).

8. Mr. L. H. Rice: *General formulation of a combinatory method used by William Emerson and others.**

The method consists in so interpreting a given problem in combinations, in terms of element, entity, subclass, and subdivision, that the problem shall take the following form: Given, a stock containing k subclasses each included in the next, the ith subclass being composed of α_i elements. Required, the number C of combinations, each separable into k mutually exclusive subdivisions, the ith subdivision being composed of β_i elements taken from among the α_i elements (or composed of β_i entities which are in one to one correspondence with β_i of the α_i elements of the stock). The answer to the problem is then apparent:

$$C = \binom{\alpha_1}{\beta_1}\binom{\alpha_2-\beta_1}{\beta_2}\binom{\alpha_3-\beta_2-\beta_1}{\beta_3}\cdots\binom{\alpha_k-\beta_{k-1}-\cdots-\beta_1}{\beta_k}.$$

9. Dr. J. L. Walsh: *A theorem on loci connected with cross-ratios.*

The following theorem, formerly proved for real λ, is now extended to non-real λ: If the respective loci of the points z_1, z_2, z_3 are circular regions, then the locus of the point z_4 determined by the constant cross-ratio $\lambda = (z_1, z_2, z_3, z_4)$ is also a circular region.

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* (1) William Emerson, *The Doctrine of Combinations, Permutations, and Composition of Quantities*, London, 1770. (2) L. Oettinger, ARCHIV FÜR MATHEMATIK, 15 (1850), p. 241, § 6. (3) L. H. Rice, *Coefficient of the general term in the expansion of a product of polynomials*, this BULLETIN, vol. 27 (1921), p. 344.

10. Mr. L. B. Robinson: *A generalization of the notion of covariants.*

Riquier has generalized the theory of complete systems. Mr. Robinson shows that a generalized complete system

$$\frac{\partial w_i}{\partial x_j} = \sum_{r=1}^{n} A_{rj} \frac{\partial w_i}{\partial t_r} + U_{ij}$$

can be utilized to generalize the notion of covariants. The generalized covariants are solutions of the above system and can be obtained by a finite number of differentiations or integrations.

11. Dr. G. A. Campbell: *Inductances of grounded circuits.*

Any network of conductors located on the surface of the earth with which it is conductively connected at any number of points will, for direct currents, have self and mutual inductances which are equal to the Neumann integral extended over the network alone. In other words, that portion of the complete Neumann integral for closed circuits which involves the return currents through the earth vanishes. The earth is assumed to be flat, of infinite extent, of unit permeability and of uniform conductivity.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The thirty-eighth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 22, 1921. Professor Lehmer presided at the earlier part of the meeting, later relieved by Professor Allardice. The total attendance was twenty-five, including the following seventeen members of the Society:

Alderton, Allardice, Bernstein, Blichfeldt, Buck, Cajori, Daus, Edwards, Haskell, Hoskins, Lehmer, Moreno, F. R. Morris, Noble, T. M. Putnam, Pauline Sperry, A. R. Williams.

The following officers were elected for the year: Chairman, Professor Allardice; Secretary, Professor Bernstein; programme committee, Professors Blichfeldt, Lehmer, Bernstein.

The dates of the next two meetings were fixed as April 8, 1922, and October 21, 1922.

Titles and abstracts of the papers read at this meeting follow. Mr. Wong was introduced by Professor Lehmer. In the absence of the authors, the papers of Professor Moritz and Professor Bell were read by title.

1. Professor Thomas Buck: *A class of asymptotic solutions of the restricted problem of three bodies.* Preliminary report.

The solutions in question are of the type considered by Poincaré, and are asymptotic to the Lagrangian equilateral triangle solutions.

2. Professor Florian Cajori: *Spanish and Portuguese symbols for "thousands."*

The Spanish symbol named *calderón,* designating "thousand," found in Spanish and Spanish-American manuscripts as well as in printed books as far back as the fifteenth century, is identified by Professor Cajori as being the same as a symbol for "thousand" used in northwestern Italy during the fifteenth century. The calderón, the author points out, is probably a modified form of the Roman symbol CIↃ for "thousand." The Portuguese *cifrão* resembles our $, and is used like the Spanish calderón, having probably the same origin.

3. Professor Florian Cajori: *History of notations of the calculus.*

The history of early calculus notations is given in detail in our histories of mathematics and books relating to Newton and Leibniz. Cantor's *Geschichte der Mathematik,* vol. 4, for the years 1759–1799, pays little attention to calculus notations of that period, and no detailed history of the nineteenth century notations exists. The present paper endeavors to cover the period since about 1740.

4. Mr. B. C. Wong: *A theorem on double points.*

The following theorem is proved: The total number D of double points on a composite curve of order n made up of r curves is greater by $r - 1$ than the maximum number D of double points on a proper curve of the same order diminished by the sum P of the deficiencies of the r curves; that is $D = D' + r - 1 - P$.

5. Professor B. A. Bernstein: *On complete independence of Hurwitz's postulates for abelian groups and fields.*

Professor Bernstein discusses the question of complete independence of each of the postulate-sets for abelian groups and fields presented by Hurwitz in the ANNALS OF MATHEMATICS, (2), vol. 15 (1913), p. 93.

6. Dr. A. R. Williams: *On finding the rate of interest on a given investment.*

The writer shows that in finding the rate of interest yielded by a redeemable security it is generally advantageous to reduce the equation to the form $i = f(i)$, where $f(i)$ is a rational function of i, whose rate of change with respect to i can be calculated if desired. It is then easy to find by trial a value of i to make the two members agree to any desired number of places. The method is self-checking, and is particularly easy of application when $f(i)$ is a decreasing function, as will be the case when the redemption price is greater than the purchase price.

7. Professor R. E. Moritz: *Anti-ratientiation; definition and fundamental relations.*

In a paper published in the AMERICAN JOURNAL, vol. 24, pp. 257–302, Professor Moritz considered certain general limiting processes of the nth order (ratientiation) which include the differentiation and quotientiation processes as special cases corresponding to $n = 0$ and $n = 1$ respectively. In the present paper the corresponding inverse processes are defined and their fundamental relations developed. The results will be embodied in a more comprehensive paper which will appear in the TÔHOKU JOURNAL.

8. Professor E. T. Bell: *Extensions of Dirichlet multiplication and Dedekind inversion.*

In this paper, which appears in this number of the BULLETIN, the author develops the theory of fields in which multiplication and division are extensions of processes defined by Cauchy, Dirichlet and Dedekind for series. The theory has been devised for the specific purpose of simplifying and correlating much of the material summarized in volume I, chapters V, X, XIX of Dickson's *History of the Theory of Numbers.*

9. Professor E. T. Bell: *A qualitative theory of singly infinite series and products.*

The object of the theory in Professor Bell's second paper is to continue that of his first and to determine under what conditions, useful in the theory of numbers, infinite series and infinite products can be manipulated to yield arithmetic theorems without any reference to their convergence or divergence.

Three kinds of multiplication of series and their inverses are discussed in detail. The third, Euler multiplication and division, is new, and connects the Dirichlet and Cauchy multiplication of series with the expansion of products of an infinite number of infinite series. Complete algebras of these processes are constructed, and each is shown to be abstractly identical with common algebra.

10. Professor E. T. Bell: *Arithmetical equivalents for a remarkable relation between theta functions.*

In a former paper, to appear shortly in the GIORNALE DI MATEMATICHE, Professor Bell gave a complete system of arithmetic equivalents for the equation of three terms in elliptic functions.

The present paper gives equivalents of a remarkable relation, due to Hermite and Enneper, which is complementary to the equation of three terms.

11. Professor E. T. Bell: *The arithmetic of the nodes and tropes on a Kummer surface.*

In a previous paper, to appear in the AMERICAN JOURNAL, Professor Bell remarked that the configuration of nodes and tropes on the Kummer surface is formally equivalent, in the sense of mathematical logic, to a system of theorems on the simultaneous representation of a pair of integers each as a sum of four squares.

This paper presents the equivalence which, for lack of space, was omitted from the other.

B. A. BERNSTEIN,
Secretary of the Section.

THE SIMPLE GROUP OF ORDER 2520

BY G. A. MILLER

Extract from a letter to F. N. Cole

If a second simple group of order $7!/2$ existed it could not be represented as a primitive group whose degree is less than twenty-one since all these primitive groups have been determined. The number of its subgroups of order 7 would be 120, for the only other divisor of $7!/2$ which is of the form $1 + 7k$ and greater than 20 is 36. It is easy to prove that such a simple group could not involve exactly 36 subgroups of order 7, as follows.

If a simple group of order $7!/2$ contained exactly 36 subgroups of order 7, it could be represented as a transitive group G on 36 letters representing the permutations of these 36 subgroups. Its subgroup G_1 composed of all its substitutions omitting one letter would be of order 70. It would therefore involve a cyclic subgroup of order 35 which would be regular, since the substitutions of order 7 would be regular. The subgroup G_1 could not be dihedral, since it could not involve negative substitutions. For the same reason the substitutions of order 2 could not transform the substitutions of order 5 into themselves and the substitutions of order 7 into their inverses. If these substitutions of order 2 could transform the substitutions of order 7 into themselves and the substitutions of order 5 into their inverses, G_1 would involve 5 conjugates of order 2. But this is impossible, since $36 \cdot 5$ is not divisible by 8.

Having proved that if the group in question existed it would contain 120 subgroups of order 7, we proceed to consider its subgroups of order 9. The number of these subgroups would be divisible by 35. In fact, if an operator of order 5 or an operator of order 7 could transform a subgroup of order 9 into itself it would be commutative with each of its operators. It was proved above that an operator of order 7 cannot be commutative with every operator of a subgroup of order 9. If an operator of order 5 were commutative with every operator of such a subgroup, the number of the subgroups of order 5 would be 56. Hence the group in question could be represented as a transitive group G of degree 56, in which G_1 would be an abelian group of order 45 involving substitutions of degree 55 and order 5. The subgroup of order 9 contained in G_1 could not have a transitive constituent of degree 9, since $8 \cdot 56$ is not divisible by 11. For a similar reason it could not have a substitution of degree 30. Hence G could not involve exactly 56 subgroups of order 5.

Since the number of subgroups of order 9 would be divisible by 35, this number would be either 70 or 280. In the latter case, the group in question could be represented as a transitive group G of degree 280, in which G_1 would be of degree 279 and of order 9. If all the substitutions besides the identity of G_1 were of degree 279, G would be of class 279 and hence could not be simple. Hence G_1 would involve at least three transitive constituents of degree 3. If a substitution of G_1 were of degree 276, it would be invariant under 4 and only 4 subgroups of order 9, and hence under a subgroup of order 36 which would contain an invariant subgroup of order 4. Hence the group under consideration could be represented as a transitive group G of degree 70 in which the G_1 of order 36 contains four subgroups of order 9.

The invariant subgroup of order 4 contained in this G_1 would be invariant under a larger group since it would be invariant under a group of order 8. This larger group would contain more than 4 subgroups of order 9 and the number of these subgroups would be divisible by 8. As this is impossible, it has been proved that if G contains 280 subgroups of order 9 the G_1 of the transitive group according to which they are transformed cannot contain a substitution of degree 276.

If this G_1 contained a substitution of degree 273, there would be a subgroup of order 63 which would involve an invariant subgroup of order 7, but this is contrary to the result obtained above. If it contained a substitution of degree 270, there would be a subgroup of order 90 which would have a quotient group of order 30. As the latter could not contain ten subgroups of order 3, this is impossible. Since no substitution of this G_1 could be of a lower degree than 270, it has been proved that the number of subgroups of order 9 contained in G is exactly 70, and that G can therefore be represented as a transitive group of degree 70 representing the transformations of its subgroups of order 9. The G_1 of this G is therefore of degree 69 and of order 36 and contains a single subgroup of order 9.

The substitutions in this subgroup of order 9 cannot all be of degree 69, since 9 does not divide 69. Hence this group of order 9 contains at least one substitution whose degree does not exceed 66. If it contains no substitution besides the identity of lower degree, it has two and only two transitive constituents of degree 3. In this case G_1 must have a transitive constituent of degree 6, since it cannot involve an invariant subgroup of order 3 and of degree less than 69. Moreover, the transitive constituent of degree 6 must be of order 36, as otherwise G_1 would involve an invariant substitution of order 2 whose degree could not exceed 36. The four subgroups of order 9 which are transformed into themselves by a substitution of order 3 and of degree 66 must therefore generate a group of order 36 in which this substitution is invariant, since a subgroup of order 18 in a transitive group of degree 6 and order 36 does not contain an invariant substitution of degree 3. It was proved above that this subgroup of order 36 contains an invariant subgroup of order 4 involving three conjugate substitutions of order 2.

As this subgroup of order 36 could have only 9 operators in common with G_1, its invariant subgroup of order 3 would be transformed into itself by 9 operators of the latter subgroup which are not found in the former subgroup of order 36. Hence we may confine our attention to the case when this invariant subgroup of order 3 would be transformed into itself by exactly 72 operators of G, and when G would involve a single set of 35 conjugate subgroups of order 3. Hence G

may be supposed to be represented as a primitive group of degree 35 and its G_1 has an invariant subgroup of order 3 whose substitutions are invariant under a G_{36} and are of degree 30. This G_{36} contains 26 substitutions of order 3 of which 10 are of degree 30 and 16 of degree 33, while its three substitutions of order 2 are of degree 28 since it has three transitive constituents, of degrees 18, 12, and 4, respectively. All of these conclusions follow from the properties of the subgroup of order 36 considered in the preceding paragraph. Each of the subgroups of order 9 has three regular constituents of order 9.

The G_1 of order 72 has also three transitive constituents, one of which is the symmetric group of degree 4. It involves 21 substitutions of order 2 and of degree 28 and 18 of order 4 and of degree 34. As all the non-invariant non-cyclic subgroups of order 4 contained in this G_1 are conjugate, this G_1 is completely determined. Its generalized dihedral subgroups of order 18 are therefore also determined. Hence the group of order 36 which contains such a dihedral group is also determined, and there is no such group besides the known simple G, which can be represented on 7 letters. It should be noted that there is a substitution of order 2 which is commutative with every substitution of the transitive constituent of degree 18 and involves only the letters of this constituent, but is not found therein. This permutes its two regular constituents of order 9 which appear in the dihedral group of order 18 noted above and it is on this account that the group of order 36 in question is completely determined. This group contains a constituent of degree 2 and together with G_1 generates G.

It has now been proved that if there were a second simple group of order $7!/2$, G_1 would involve substitutions whose degree would be less than 66. As the subgroup of order 9 in G_1 could not contain exactly five transitive constituents of degree 3 it would have to involve at least 8 such constituents. If G_1 contained a substitution of order 3 and of degree 63, the seven subgroups of order 9 which would be transformed into themselves by this substitution would generate a group in which this substitution is invariant, and hence this group would contain exactly 7 subgroups of order 9. Its order would therefore be 63, 126, or 252. This is impossible as each of these groups would involve only one subgroup of order 7 since each of its subgroups of order 7 would be transformed into itself by at least 21 substitutions.

It remains only to consider the case when G_1 would contain a substitution of order 3 and of degree 60 without involving such a substitution of degree 63. The order of the group formed by all the substitutions of G which would be commutative with this substitution of order 3 would be 90. This group of order 90 would transform its ten subgroups of order 9 according to a transitive group of order 30 and of degree 10. Since this transitive group does not exist,* we have arrived at nothing but contradictions by assuming the existence of a second simple group of order $7!/2$ and hence such a group is actually proved to be non-existent.

THE UNIVERSITY OF ILLINOIS.

A THEOREM OF OSCILLATION

BY W. E. MILNE

In an investigation of the oscillations of aerial bombs a need was found for the following proposition. Both the theorem and its proof are modelled after a similar theorem and proof by Osgood.†

THEOREM. *Let* $\varphi(t)$ *be positive, continuous, monotonically increasing, and bounded in the interval* $T \leqq t < \infty$, *and let* m *and* M *be two positive constants such that* $m < \varphi(t) < M$ *for* $t > T$. *Let* $f(y)$ *be an odd, monotonically increasing function, satisfying the Lipschitz condition*

$$|f(y_1) - f(y_2)| < K|y_1 - y_2|, \qquad K > 0,$$

in an interval $-a \leqq y \leqq +a$, $a > 0$. *Let* y *be a solution of the differential equation*

$$\frac{d^2y}{dt^2} + \varphi(t)f(y) = 0 \tag{1}$$

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* Cf. F. N. Cole, QUARTERLY JOURNAL, vol. 27 (1895), p. 40.

† This BULLETIN, vol. 25 (1919), pp. 216–221.

subject to the conditions

$$(2)\quad \frac{dy}{dt} = 0,\ y = y_1,\ |y_1| < a,\ f(y_1) \neq 0,\ \textit{when}\ \ t = t_1 > T.$$

Then y oscillates an infinite number of times in the interval $t_1 < t < +\infty$ *and the amplitudes decrease monotonically but do not approach zero.*

Proof: Let us extend* the definition of $f(y)$ by the formulas

$$f(y) = f(a), \qquad \text{when} \qquad y > a,$$
$$f(y) = f(-a), \qquad \text{when} \qquad y < -a.$$

The function so extended satisfies the Lipschitz condition.

With the hypotheses thus extended, there exists† a unique function $y(t)$, continuous together with its first two derivatives, which satisfies (1) and (2) in the interval $t_1 \leqq t < \infty$. Now consider the case in which y_1 is positive. Then, at t_1, d^2y/dt^2 is negative and remains negative as long as y is positive. Since

$$v = \frac{dy}{dt} = \int_{t_1}^{t} \frac{d^2y}{dt^2}\,dt, \qquad t > t_1,$$

we see that v is negative as long as y is positive. Therefore the graph of $y(t)$ as a function of t is concave downward with negative slope to the right of t_1, and therefore must cut the axis at a finite point $t_1' > t_1$. Let v_1 be the corresponding value of v. Now multiply (1) by $2dy$ and integrate, obtaining

$$(3)\qquad v^2 = -2\int_{y_1}^{y} \varphi(t) f(y)\,dy.$$

At t_1' this becomes

$$v_1^2 = 2\int_0^{y_1} \varphi(t) f(y)\,dy.$$

Since in the interval $t_1 \leqq t \leqq t_1'$ we have by hypothesis $\varphi(t_1) \leqq \varphi(t) \leqq \varphi(t_1')$, it follows that

$$v_1^2 \leqq 2\varphi(t_1') \int_0^{y_1} f(y)\,dy,$$

$$v_1^2 \geqq 2\varphi(t_1) \int_0^{y_1} f(y)\,dy.$$

Now let

$$\int_0^{y} f(y)\,dy = F(y).$$

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* This extension is made for convenience in establishing the existence of the solution. Actually the definition of $f(y)$ outside the interval $-a \leqq y \leqq a$ is immaterial.

† Bliss, PRINCETON COLLOQUIUM, p. 93.

Then $F(y)$ is even and continuous for $|y| < a$, is monotonically increasing in the interval $0 < y < a$, and vanishes at the origin. With this notation the above inequalities become

$$\left\{\begin{array}{l} v_1^2 \leqq 2\varphi(t_1')F(y_1), \\ v_1^2 \geqq 2\varphi(t_1)F(y_1). \end{array}\right. \tag{4}$$

At t_1' v is negative; hence, immediately to the right of t_1' y is negative, and therefore d^2y/dt^2 is positive. Moreover as long as v is negative, d^2y/dt^2 is monotonically increasing, as equation (1) shows. Then since

$$v = v_1 + \int_{t_1'}^{t} \frac{d^2y}{dt^2}dt,$$

it is clear that v must vanish for a finite value of t, $t = t_2 > t_1'$. Let the corresponding value of y be y_2. Now from (3)

$$v_1^2 = 2\int_0^{y_2}\varphi(t)f(y)dy,$$

whence as in the preceding case

$$\left\{\begin{array}{l} v_1^2 \leqq 2\varphi(t_2)F(y_2), \\ v_1^2 \geqq 2\varphi(t_1')F(y_2). \end{array}\right. \tag{5}$$

From (4) and (5) we get

$$F(y_1) \geqq F(y_2), \qquad \text{or} \qquad |y_1| \geqq |y_2|,$$
$$\varphi(t_1)F(y_1) \leqq \varphi(t_2)F(y_2).$$

A similar argument leads to the same results when y_1 is negative.

Starting now with the conditions $dy/dt = 0$, $y = y_2$, when $t = t_2$, we may repeat the entire argument and obtain $|y_2| \geqq |y_3|$, $\varphi(t_2)F(y_2) \leqq \varphi(t_3)F(y_3)$, and in general

$$|y_n| \geqq |y_{n+1}|, \tag{6}$$

and

$$\varphi(t_n)F(y_n) \leqq \varphi(t_{n+1})F(y_{n+1}), \tag{7}$$

where t_n is the nth value of t (beginning with t_1) for which $dy/dt = 0$, and y_n is the corresponding value of y.

The quantities y_n are the amplitudes of the successive oscillations. Hence (6) proves that the amplitudes decrease monotonically. From (7), together with the hypotheses regarding $\varphi(t)$, it may be shown that $F(y_n) \geqq (m/M)F(y_1)$, which proves that the amplitudes do not approach zero.

The University of Oregon.

A TWO-WAY INFINITE SERIES FOR LEBESGUE INTEGRALS*

BY M. B. PORTER

1. *The Defining Series.* Let $F(x)$ denote any positive measurable function of n variables defined over a measurable point set δ. We shall use the sign δ to denote either a point set or its measure, and the sign† δ^* to denote δ diminished by a nul-set or to indicate that a property holds *almost everywhere* (*presque partout*) over δ.

Now write

$$(1) \qquad \Phi(\delta) = \sum_{i\to-\infty}^{\infty} 2^i\phi_i(\delta) = \operatorname*{Lim}_{m\to\infty}\sum_{i\to-\infty}^{m} 2^i\phi_i(\delta),$$

where $\phi_i(\delta)$ is the measure of all those points E_i of δ for which $F(x)$, expressed in the binary scale, has unity in the ith place, it being understood that $F(x)$ shall be always expressed in closed form when possible (or else never so expressed), in order that the representation shall be unique. If $F(x)$ is limited, it is evident that (1) will converge uniformly over δ. If $F(x)$ is *not* limited, (1) may still converge; if it does, the convergence is necessarily uniform. In that case $F(x)$ is summable over δ, and we shall indicate the sum of the series by $\int F(x)$, the Lebesgue integral of F over δ. The integral is definite or indefinite according as we regard δ as fixed or as variable.

In case $F(x)$ is not always positive, set, as usual,

$$\begin{aligned} \phi(x) &= F(x) \text{ when } F(x) \geqq 0, & \psi(x) &= F(x) \text{ when } F(x) < 0,\\ \phi(x) &= 0 \text{ when } F(x) < 0, & \psi(x) &= 0 \text{ when } F(x) \geqq 0; \end{aligned}$$

then

$$(2) \qquad \Phi(\delta) = \sum_{-\infty}^{\infty} 2^i\phi_i(\delta) - \sum_{-\infty}^{\infty} 2^j\psi_j(\delta),$$

and this difference may converge even when the separate series diverge, if i and j become simultaneously infinite in a suitable manner.

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* Presented to the Society, September 9, 1921.

† Cf. Pierpont's useful notation, *Functions of a Real Variable*, vol. 1, p. 119.

Thus, if

$$F(x) = \frac{d}{dx}\left(x \sin \frac{1}{x}\right), \qquad x \neq 0,$$

$F(x)$ is not summable at $x = 0$; but if from the interval (a, x), $a < 0$, $x > 0$, we remove the interval 2ϵ symmetric about the origin, the integral (2) will exist over the remaining domain and will be equal to $x \sin (1/x) - a \sin (1/a)$. A similar procedure can be applied where $F(x)$ has a countable set of non-summable points, and is the derivative of a continuous function.

Let η denote a set of infinitesimal hyperspheres about x, a point of δ, as center. We shall show that

$$\text{(3)} \qquad \operatorname*{Lim}_{\eta \to 0} \frac{\Phi(\eta)}{\eta}$$

exists over δ^*. Since this is a point-function we shall use the argument x in the limit.

Lebesgue has shown in his metric-density theorem that for the measure functions ϕ_i the limit (3) exists almost everywhere, and that†

$$\text{(4)} \qquad \begin{aligned} \phi_i'(x) &= 1 \quad \text{over} \quad E_i^*, \\ \phi_i'(x) &= 0 \quad \text{over} \quad C^*E_i, \end{aligned}$$

which in one dimension implies the existence of the derivative in the ordinary sense of the word over E_i^* and $C_i^*E_i$ with the values given in (4).

2. *Special Theorems.* From (4) we have the following theorems.

THEOREM A. *For any measurable function $F(x)$,*

$$F'(x) = \sum_{-\infty}^{\infty} 2^i \phi_i'(x) - \sum_{-\infty}^{\infty} 2^j \psi_j'(x),$$

both series converging over that part of δ^* where $F(x)$ is finite.

THEOREM B. *In case $F(x)$ is limited, $\int_\delta F(x)$ exists and its derivative is*

$$\int_\delta' F(x) = \sum_{-\infty}^{m} 2^i \phi'(x) - \sum_{-\infty}^{m} 2^j \psi'(x) = F(x)$$

over δ^*, the series converging uniformly over δ.

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† Various proofs of this theorem have been given, e.g. a simple proof based on Vitali's covering theorem by de la Vallée Poussin, *Cours d'Analyse*, vol. 2, p. 110 et seq. See also Hobson, *Real Variables*, vol. 2, p. 178 et seq.

If $\psi(\delta)$ denote a function with a bounded upper symmetrical derived number $D^+\psi(x)$, then

$$f(\delta) = \int_\delta D^+\psi(x)$$

exists; and, since $f'(x) = D^+\psi(x)$ over δ^*, $f(\delta)$ and $\psi(\delta)$ differ* by a constant. In one dimension, this is Lebesgue's theorem that *a monotone increasing function with bounded derived numbers has a derivative almost everywhere.*

Theorem B can be extended by means of Vitali's covering theorem to any summable function, but we shall not stop to give here the details of the proof. The convergence of the series will be uniform over a set δ_1 differing from δ by as little as we please.

3. *Fundamental Theorems.* It will be readily seen that the following fundamental theorems follow at once from (1).

(1) The mean value theorem for integrals.

(2) The theorem that $\int_{\delta_1} f + \int_{\delta_2} f = \int_{\delta_1+\delta_2} f$. where δ_1 and δ_2 have only a nul-set in common.

(3) The theorem that $\int_\delta [f_1 + f_2] = \int_\delta f_1 + \int_\delta f_2$, proved by truncating f_1 and f_2, deriving, and then using Scheeffer's theorem.

(4) The theorem that $\lim\limits_{n\to\infty} \int_\delta f_n(x) = \int_\delta \lim\limits_{n\to\infty} f_n(x)$.

In conclusion we shall make some applications of Theorem A which will show its use in dealing with measurable functions.

4. *Application to Almost Everywhere Finite Measurable Functions.* If, in Theorem A, we cover each E_i set by a finite set of non-overlapping cells C_i such that

$$|\text{meas } E_i - \Sigma C_i| < \frac{\epsilon}{2^i},$$

and if we define a function equal to *one* over the C_i's and to *zero* over the complementary cells, we shall have a discontinuous function which can be made continuous by *trimming* the C_i cells or their complements and using suitable *connective tissue.* Thus we have the following theorem.

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* In one dimension, this follows from Lebesgue's extension of Scheeffer's theorem, Lebesgue, *Leçons sur l'Intégration*, p. 79.

THEOREM C. *Any almost everywhere finite measurable function is equal to a continuous function save over a set of points of arbitrarily small measure* $\bar{\epsilon}$.*

If we denote this continuous function by $C(x)$, we have $\int_\delta F(x) = R\int_{\bar{\delta}} C(x) + \bar{\bar{\epsilon}}$ when δ and $\bar{\delta}$ differ by as little as we please, where $R\int_\delta$ denotes the Riemann integral.

5. *Egoroff's Theorem.* From § 4, we can deduce Egoroff's theorem:† *If a sequence of measurable functions* $F_i(x)$ *converges to a limit* $F(x)$ *over* δ *it will converge uniformly save over a subset of arbitrarily small measure.*

6. *Lusin's Extension of Weierstrass' Theorem.* In § 4, by using a set of $\bar{\epsilon}$'s converging to zero, we have the theorem: *Any in general finite measurable function is the limit almost everywhere of a sequence of continuous functions.* Hence we have also Lusin's‡ extension of Weierstrass' theorem, viz.: *Any measurable function is almost everywhere the limit of a sequence of polynomials.*

7. *Approximately Continuous Functions.* Making use of Denjoy's definition of *approximately continuous functions,*§ we have the theorem: *A measurable almost everywhere finite function is approximately continuous save for a nul-set.*

By means of Vitali's covering theorem it is easy to show that *any almost everywhere approximately continuous function is measurable.* Hence we can say that the R-integrable functions are almost everywhere continuous and the L-integrable functions are almost everywhere approximately continuous, when it is understood, of course, that the functions are limited.

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* Cf. Lusin, COMPTES RENDUS, June 17, 1912.

† COMPTES RENDUS, Jan. 30, 1911.

‡ Lusin, COMPTES RENDUS, loc. cit.

§ A function $F(x)$ is approximately continuous at x when it is continuous over a point set of metric density one at x. BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 43, p. 165; or Hobson, vol. 1, p. 295.

NOTE ON EULER'S φ–FUNCTION

BY R. D. CARMICHAEL

Two correspondents have recently called my attention to the fact that the supposed proof of the following theorem, which I gave some years ago,* is not adequate:

THEOREM I. *For a given number n, the equation* $\varphi(x) = n$ *either has no solution or it has at least two solutions.*

So far I have been unable to supply a proof of the theorem, though it seems probable that it is correct. I am therefore compelled to allow it to stand in the status of a conjectured or empirical theorem.

Let us examine the hypothesis that there exists a value ν of n such that $\varphi(x) = \nu$ has one and just one solution. It is easy to derive certain necessary properties of x. In the first place, x is even, since otherwise $2x$ would also be such that $\varphi(2x) = \nu$. Again, x is divisible by 4, since otherwise $\varphi(x/2)$ would be equal to ν. Let us then denote the value of x by $4s$. We shall prove the following theorem.

THEOREM II. *If* $4s$ *has the factor* $p_0^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, *where* p_0 $(= 2)$, p_1, p_2, $\cdots$, p_k *are distinct prime numbers, and if the quotient of* $4s$ *by this factor is prime to the factor, and if* $p_0^{\gamma_0}p_1^{\gamma_1} \cdots p_k^{\gamma_k} + 1$ *is a prime number* q, *where for a given* i, $0 < \gamma_i < \alpha_i$, *then* $4s$ *has the factor* q^2.

The proof is almost immediate. For we have

$$\varphi(2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \varphi(2^{\alpha_0-\gamma_0}p_1^{\alpha_1-\gamma_1}p_2^{\alpha_2-\gamma_2} \cdots p_k^{\alpha_k-\gamma_k}q),$$

so that we should have two solutions of the equation $\varphi(4s) = \nu$ unless s contains the factor q. Similarly, it may be shown that s contains the factor q^2, since otherwise the first power of q could be omitted by appropriately raising certain (or all) of the exponents on the p's without affecting the value of the φ-function.

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* This BULLETIN, vol. 13 (1907), p. 241. The theorem is also stated as an exercise in my *Theory of Numbers*, p. 36; it was its presence here that led each correspondent to the discovery of the error.

In the same way we may prove the following theorem.

THEOREM III. *If s is divisible by a prime of the form $2^k + 1$ it is divisible by the square of this prime.*

For if $4s = 2^{\alpha_0}(2^k + 1)s_1$, where s_1 is prime to $2(2^k + 1)$, we have

$$\varphi(4s) = 2^{\;{}_0-1}2^k \cdot \varphi(s_1) = \varphi(2^{k+\alpha_0}s_1),$$

contrary to the hypothesis that the solution is unique.

From Theorem II, it follows that s has the factor 3^2, thence that it has the factor 7^2, and thence that it has the factor 43^2.

Now suppose that s does not have the factor 3^3. Then since $\varphi(2^{\alpha_0} \cdot 3^2) = 2^{\alpha_0} \cdot 3 = \varphi(2^{\alpha_0 - 1} \cdot 13)$, if $\alpha_0 > 1$, it is readily shown that s has the factor 13^2. For this case, then, $4s$ has the factor $2^2 \cdot 3^2 \cdot 7^2 \cdot 13^2 \cdot 43^2$. We may now apply Theorem II successively to show that $4s$ has the factors 79^2, 547^2, 3319^2, 1854763^2. It appears possible to determine in the same way still other factors of $4s$. Those obtained are sufficient to show that $4s$ has at least 38 digits.

If $4s$ contains the factor 3^3, then it follows from Theorem II that it has the factors 19^2, 127^2. Then $4s$ has the factor $2^2 \cdot 3^3 \cdot 7^2 \cdot 19^2 \cdot 43^2 \cdot 127^2$. Applying Theorem II successively we may show that $4s$ has the factors 2287^2, 101347^2, 304039^2. It appears possible to determine in the same way still other factors of $4s$. Thence it is easy to see that $4s$ in this case has at least 41 digits.

Hence, if the empirical Theorem I is not valid, the unique solution x (which would then exist for at least one value of n) must contain at least 38 digits. Moreover, it is easy to increase this number of necessary digits for these exceptional solutions. Hence it seems probable that the empirical theorem is true.

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EXTENSIONS OF DIRICHLET MULTIPLICATION AND DEDEKIND INVERSION*

BY E. T. BELL

1. *Introduction.* With Landau,† let us define *Dirichlet multiplication* to be the following formal process of combining two sequences α_n, β_n $(n = 1, 2, \cdots)$ to form a third sequence γ_n, $\gamma_n = \Sigma\alpha_l\beta_m$, the sum extending to all integers l, $m > 0$ such that $lm = n$. Henceforth, unless otherwise stated, we assume the four rational algebraic operations to be purely formal.‡ When any of these operations in relation to a given system of elements have special interpretations, examples of which are noted in a moment, they will be called *specific.* Restating Landau's definition, let us denote by α_n, β_n $(n = 1, 2, \cdots)$ two classes of elements which are such that the product of any element of one class by an element of the other has a unique significance, and likewise for any sum of such formal products. Then the *Dirichlet product* of these classes is the class $(\alpha, \beta)_n$ $(n = 1, 2, \cdots)$, where $(\alpha, \beta)_n = \Sigma\alpha_l\beta_m$ $(lm = n)$.

When the α's and β's are known, the process of determining the φ's from the relation $(\alpha, \varphi)_n = \beta_n$ for n an arbitrary integer > 0 will be called *Dedekind inversion.* It will be shown that the extension of this which we have in view includes Dedekind's inversion in the theory of numbers.§

The interpretations of the general theorems in a specific case will depend only upon the meanings assigned to the elements and to the rational operations upon those elements. Thus, if the elements are numbers and the rational operations are as in arithmetic, the interpretation is obvious; if the elements are classes, multiplication and addition are logical,

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* Presented to the Society, October 22, 1921.

† Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 2, p. 671. We replace his series by sequences.

‡ As in the customary definitions of an abstract field, cf. Dickson, *Linear Groups*, pp. 5–6. When some of the operations in a field are specific, the field will be called special. See also § 6.

§ The inversion is usually attributed to Dedekind, although it was published simultaneously by Liouville. Cf. Dickson's *History*, vol.1, p. 441.

their inverses are not defined, and the theorems apply only to all sums and products derived from products which are themselves obtained by multiplications from a primary set;* if the elements are single-valued functions (or if they are associates, as presently defined), the important cases for the arithmetic of integers or of ideals, the interpretations of the rational operations are as developed later in this paper. By means of the last a great mass of the material which is summarized in Chapters V, X, XIX of volume I of Dickson's *History of the Theory of Numbers*, and which is not most naturally derived from the elements of elliptic functions, can be simultaneously restated in much condensed form, unified, proved almost at a glance, and extended in many directions. This, the most immediate application of what follows, will be discussed elsewhere.†

2. *Definitions*. (i) Let A' be a class of n' independent elements x_a $(a = 1, 2, \cdots, n')$ subject only to the commutative and associative laws of multiplication. By definition the zero powers of all elements in A' are equal, and each is equal to the multiplicative unit e, thus $x_a^0 = e$ $(a = 1, 2, \cdots, n')$, $e^2 = e$, etc.; and n' may be finite or infinite. From the elements of A' we form the class A of all products of the type $x_a^\alpha x_b^\beta \cdots x_c^\gamma \equiv \Pi x_a^\alpha$, where $a, b, \cdots, c$ are different members of the set $1, 2, \cdots, n'$, and $\alpha, \beta, \cdots, \gamma$ are integers $\geqq 0$, and denote this typical product by x. Products differing only by unit factors (powers of e) are equal. Hence if n' is finite, A may consist of either a finite or infinite number n of distinct products. The class A evidently includes all the members of A'. The members of A' are called the *primary elements* of A. Elements of A' are called the *derived elements*, and x is a typical derived element. From these definitions we may con-

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* Subtraction and division are omitted for the reasons given by Whitehead, *Universal Algebra*, vol. 1, p. 82.

† Another application of importance for arithmetic, which has been developed in detail, is to the case in which the elements are sequences. A brief account of results for elliptic functions cited above which involve Bernoullian functions appeared recently in the MESSENGER OF MATHEMATICS; generalizations for certain of the other divisor theorems depending on elliptic functions will be published later.

sider the primary elements to be independent generators of an abelian group A whose identity is e. The order of the group is n, which may be finite or infinite. When A' is the class of prime numbers and multiplication is as in arithmetic, the resulting theory is called the numerical case.

(ii) An *associate* of an element in A is anything that has a unique significance in terms of elements of A, when the element, called the *argument* of the associate, is assigned. It is assumed that associates admit of unambiguous combination by formal rational operations (§ 1) with associates. If A be replaced throughout in what precedes by A', the class of primary elements of A, there is defined a primary associate. When the argument is a derived element, the associate is called derived. This distinction is made because all that follows is based upon associates that need exist only when their arguments are primary elements; derived associates are constructed (in (iv)) from the primaries.*

(iii) The notation being as in (i), the $\nu(x) = (\alpha + 1)(\beta + 1) \cdots (\gamma + 1)$ derived elements x' given by

$$x' = x_a^{\alpha'} x_b^{\beta'} \cdots x_c^{\gamma'} \quad (0 \leqq \alpha' \leqq \alpha,\ 0 \leqq \beta' \leqq \beta,\ \cdots,\ 0 \leqq \gamma' \leqq \gamma)$$

are by definition the divisors of the typical derived element x. In the product $x \equiv \Pi x_a^{\alpha}$ of powers of distinct primary elements, corresponding Latin and Greek letters (a, α), (b, β), $\cdots$, (c, γ) are associated with a given primary element, thus

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* Since it seems not to be customary except in mathematical logic (*Principia Mathematica*, vol. 1, pp. 15–18) to speak of either values or functions apart from a numerical context, we use *associate* instead of *single-valued function*, although the latter would be justified by the current use of *propositional function*. For special interpretations of the several parts of (ii) the definition of associate degenerates as follows to that of single-valued function of an integer > 0. Let $\varphi(x)$ be an associate having the integral non-zero argument x. Give the parts of (ii) the following specific meanings, which obviously are legitimate: the rational operations are as in ordinary algebra; $A \equiv$ the class of integers > 0; the phrase "when the element is assigned" $\equiv$ *when x lies in the interval (a, b) of A;* "has a unique significance in terms of elements of A" $\equiv$ *has a single definite value corresponding to each x $(a \leqq x \leqq b)$, no matter in which form the correspondence is specified.* With these interpretations, $\varphi(x)$ is a single-valued function of x in the interval (a, b). Cf. Dirichlet's definition (*Werke*, vol. 1, p. 135), also Hobson, *Theory of Functions of a Real Variable*, 1st edition, p. 216) for the analogous definition when x, (a, b) are continuous.

$x_a^\alpha, x_b^\beta, \cdots, x_c^\gamma$. This convention is maintained throughout when the exponent is an arbitrary positive integer.

(iv) With each positive integral or zero power x_d^i of any primary element of x_d are associated specific primary associates $\varphi_i(x_d), \psi_i(x_d), \cdots, \mu_i(x_d), \cdots$ of that element, the suffix i of the primary associate being equal in each case to the exponent of the power. For example the φ-primary associates of x_a^0, $x_a^1, x_a^2, \cdots$ are $\varphi_0(x_a), \varphi_1(x_a), \varphi_2(x_a), \cdots$. By definition all primary associates of e are equal, and each is denoted by ϵ. Hence since by (i) $x_a^0 = x_b^0 = \cdots e$, we have $\varphi_0(x_a) = \varphi_0(x_b) = \cdots = \epsilon$; $\mu_0(x_a) = \mu_0(x_b) = \cdots = \epsilon$, etc. We call ϵ the *unit associate. It is assumed that, with respect to any associate, ϵ has all the multiplicative properties of unity in arithmetic.* Beyond the trivial one that they both have the same argument x_a, it is not assumed (without explicit statement of the assumption) that there is any relation whatever between $\varphi_m(x_a)$, and $\varphi_n(x_a)$ (m, n integers $\geqq 0$, $m \neq n$); nor is it assumed that primary associates having different suffixes are necessarily distinct.*

From the definition in (ii), it follows that, for $x = \Pi x_a^\alpha$ as in (i),

$$\Pi\varphi_\alpha(x_a) \equiv \varphi_\alpha(x_a)\varphi_\beta(x_b) \cdots \varphi_\gamma(x_c)$$

is an associate of x. We call this a *derived associate,* and we shall denote it by φ_x. The derived associate $\varphi_{x'}$ of the divisor x' of x is called an *associate divisor* of φ_x. Similarly, starting with the primary associates $\psi_\alpha(x_a), \psi_\beta(x_b), \cdots, \psi_\gamma(x_c)$, we define $1 + \nu(x)$ derived associates $\psi_x, \psi_{x'}$; and so on for $\chi_x, \chi_{x'}; \lambda_x, \lambda_{x'}; \cdots; \mu_x, \mu_{x'}$. Thus $\mu_x = \Pi\mu_\alpha(x_a)$. Henceforth we shall drop the adjectives *primary* and *derived,* and refer only to associates, since the notations $\varphi_\alpha(x_a)$, φ_x suffice without qualifications.

3. *Relation with Group Characters.* Before proceeding, we emphasize the remarks made in § 1 regarding the interpretations of results. This has particular reference to the assump-

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* For example in the special case of associate $\equiv$ single-valued function, we may have $\varphi_\alpha(x_a) = \sin \pi x_a$, $\varphi_\beta(x_b) = \sin \pi x_b$; namely each is the sine of π times its argument.

tion in § 2 (ii), to which we shall add some comments in § 6. The development is necessarily abstract, because in the applications it is essential that proofs be based upon considerations that are independent of convergence, or indeed of any infinite process, and because each general theorem is susceptible of many specific interpretations. (Cf. § 1.) It may assist the reader to consult § 8 occasionally. We insist, however, that nothing in § 8 is assumed in any of the proofs, all of which are immediate consequences of the definitions, except that those after Theorem XIV depend also upon the assumption that an arithmetic system exists. This assumption is satisfied by the class of all rational functions in any number n of independent variables with integral coefficients, which for $n = 1, 2$ is the important case arithmetically. We have assumed that for each primary element there exists at least one associate. This can be satisfied, for example, by taking the associate equal to the element. Hence the theory as developed relates to objects that exist.

In addition to possible interpretations mentioned in § 1 we note one here which is of particular interest.* If we impose the restrictions $f_2(x_a) = [f_1(x_a)]^2$ ($f = \varphi, \psi, \chi, \cdots, \lambda, \mu, \cdots$) upon the associates, they can be interpreted as group characters.†

4. *Extended Dirichlet Product.* If the derived elements x_1, x_2 (§ 2 (i)) have in common only unit factors (powers of e), they are called *relatively distinct* (the analogue of *relatively prime*).

THEOREM I. *If* x_1, x_2 *are relatively distinct,* $\varphi_{x_1}\varphi_{x_2} = \varphi_{x_1x_2}$.

Next, let x_1, x_2 be *any* pair of divisors (§ 2 (iii)) of x that are such that $x_1x_2 = x$; form $\varphi_{x_1}\psi_{x_2}$ for each pair (x_1, x_2); take the sum of all such products for all possible pairs (x_1, x_2); and denote that sum by $(\varphi, \psi)_x$. It is obvious, by § 2 (ii), (iv), that this sum is an associate of x, and hence the process may be repeated for $(\varphi, \psi)_x$ and χ_x as initial associates instead of φ_x, ψ_x; denote the result by $((\varphi, \psi), \chi)_x$, or briefly by $(\varphi, \psi, \chi)_x$. Continuing thus, starting with m associates $\varphi_x, \psi_x, \chi_x, \cdots, \lambda_x, \mu_x$, taken in this order, we reach finally an associate $(\varphi, \psi, \chi, \cdots, \lambda, \mu)_x$ of x, which is called the *Dirichlet product* of the original m associates, and the original m associates are called the *primitive divisors* of the extended product.

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* This was pointed out by Professor Dickson. I had not noticed it, since I was developing the theory with reference to its applications to sequences, classes, and the multiplicative properties of numbers and ideals.

† For the case of abelian groups, cf. Weber, *Algebra*, 2d edition, vol. 2, p. 43; for an exposition of the general case, cf. Dickson, ANNALS OF MATHEMATICS, (2), vol. 4, 1902, pp. 25–49.

THEOREM II. *The extended product* $(\varphi, \psi, \cdots, \mu)_x$ *is a symmetric function of* $\varphi, \psi, \cdots, \mu$. *In other words, extended multiplication is associative and commutative.*

This being a result of great power in applications, we shall examine it more closely. Let $x_1, x_2, \cdots, x_m$ denote divisors of x such that $x = x_1x_2 \cdots x_m$. Then we have

$$(\varphi, \psi, \cdots, \mu)_x = \Sigma\varphi_{x_1}\psi_{x_2} \cdots \mu_{x_m},$$

where the sum extends to all sets $(x_1, x_2, \cdots, x_m)$. The m functions in the typical term on the right are associate divisors (§ 2 (iv)) of $\varphi_x, \psi_x, \cdots, \mu_x$, respectively. From the mode of formation it is evident that the sum (i.e., the extended Dirichlet product) is invariant under permutations of some or all of $x_1, x_2, x_3, \cdots, x_m$. This sum is an associate of x, say $F_x^{(m)}$, which is uniquely determined whenever x and the primitive divisors $\varphi_x, \psi_x, \cdots, \mu_x$ of $(\varphi, \psi, \cdots, \mu)_x \equiv F_x^{(m)}$ are assigned; in particular it is uniquely known when x is replaced by any one of its divisors x'. In the same way, starting with any n associates $\alpha_x, \beta_x, \cdots, \gamma_x$, we arrive at $G_x^{(n)}$, and continuing thus we obtain a system of r such extended products $F_x^{(m)}, G_x^{(n)}, \cdots, H_x^{(t)}$, there being r letters $m, n, \cdots, t$. Let $m + n + \cdots + t = s$, and suppose that the s associates upon which the r extended products are constructed are $\alpha_x, \beta_x, \cdots, \omega_x$, these being not necessarily distinct. Then we have

$$(F^{(m)}, G^{(n)}, \cdots, H^{(t)})_x = (\alpha, \beta, \cdots, \omega)_x.$$

Except when $m = n = \cdots = t = 1$, the primitive divisors $F_x^{(m)}, G_x^{(n)}, \cdots, H_x^{(t)}$ and $\alpha_x, \beta_x, \cdots, \omega_x$ of these extended products are not the same. On the left, the extended product refers to resolutions of x into r divisors which in turn are resolved into $m, n, \cdots, t$ divisors respectively, the resolutions in both cases being all possible of their respective kinds; on the right, the resolution is into all possible sets of s divisors. The identity of the formal sums for these distinct methods of resolving x is the interpretation of the associative and commutative properties of extended multiplication. The significance of Theorem II in the numerical case was stated in a former paper.*

In order to see in the numerical case that $(\varphi, \psi)_x$ is equivalent to $(\alpha, \beta)_n$ of § 1 when φ_x, ψ_x, and the α's and β's are single-valued functions of integers, take as primary elements the natural primes, choose $\varphi_\rho(x_r) = \varphi'(x_r{}^\rho)$ where φ' is single-valued and such that $\varphi'(mn) = \varphi'(m)\varphi'(n)$ when m and n are relatively prime integers greater than zero. In the same way, define $\psi_\rho(x_r) = \psi'(x_r{}^\rho)$, and finally write $\varphi'(x)$, $\psi'(x) \equiv \varphi_x'$, ψ_x'. Then for α, β, n in the numerical case equal respectively φ', ψ', x, as above, we have $(\varphi', \psi')_x = (\alpha, \beta)_n$.

Next, the sum of the associates $\chi_x, \cdots, \lambda_x, \mu_x$ is obviously an associate, σ_x, of x, and it is easily seen that

$$(\varphi, \sigma)_x = (\varphi, \chi)_x + \cdots + (\varphi, \lambda)_x + (\varphi, \mu)_x.$$

THEOREM III. *Extended Dirichlet multiplication is distributive with respect to (formal) addition.*

5. *Extended Dedekind Inversion.* The unit associate ϵ was defined in § 2 (iv). We now introduce the *associate unit* $\eta_x \equiv \eta$, whose definition is as follows: $\eta_x = 0$ (the multiplicative zero) when $x \neq e$ (the multiplicative unit of A); $\eta_x = \epsilon$ (the unit associate) when $x = e$.

THEOREM IV. *For each associate φ_x there exists a unique associate φ_x' such that $(\varphi, \varphi')_x = \eta$.*

We shall call φ_x' the *reciprocal* of φ_x, and, by an obvious algebraic analogy, we shall write $\varphi_x' = \eta/\varphi_x$, $\varphi_x = \eta/\varphi_x'$. We have at once, ψ_x being any associate, $(\psi, \eta)_x = \psi_x$; and hence we have the following theorem.

THEOREM V. *The unit with respect to extended Dirichlet multiplication is η.*

To prove Theorem IV, we shall exhibit φ_x' explicitly in terms of φ-associates. We observe first that if $x = \Pi x_a^\alpha$ as usual, Theorem I enables us to state that φ_x' is equal to the formal product of the associates φ_ξ' for $\xi = x_a^\alpha, x_b^\beta, \cdots, x_c^\gamma$. Hence it is sufficient to determine φ_x' when $x = x_a^\alpha$, $\alpha > 0$. By the convention of § 2 (iv), the φ-primary associates of x_a^1, x_a^2, $\cdots$, x_a^α are respectively $\varphi_1(x_a)$, $\varphi_2(x_a)$, $\cdots$, $\varphi_\alpha(x_a)$. For $x = x_a^\alpha$, write φ_x' in the form $\varphi_\alpha'(x_a)$. Then it can be verified without difficulty that $\varphi_\alpha'(x_x)$ is equal to the determinant

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* This BULLETIN, vol. 27, p. 274, § 2.

$$\left(-\frac{1}{\epsilon}\right)^\alpha \begin{vmatrix} \varphi_1(x_a) & \epsilon & 0 & \cdots & 0 \\ \varphi_2(x_a) & \varphi_1(x_a) & \epsilon & \cdots & 0 \\ \varphi_3(x_a) & \varphi_2(x_a) & \varphi_1(x_a) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{\alpha-1}(x_a) & \varphi_{\alpha-2}(x_a) & \varphi_{\alpha-3}(x_a) & \cdots & \epsilon \\ \varphi_\alpha(x_a) & \varphi_{\alpha-1}(x_a) & \varphi_{\alpha-2}(x_a) & \cdots & \varphi_1(x_a) \end{vmatrix}.$$

It is easy to prove this also by mathematical induction from $\alpha - 1$ to α, $\alpha > 1$. Next, φ_x' is written down as the formal product of $\varphi_\alpha'(x_a)$, $\varphi_\beta'(x_b)$, $\cdots$, $\varphi_\gamma'(x_c)$, which are obtained from the above on replacing (α, a) in turn by (α, a), (β, b), $\cdots$, (γ, c). It may be mentioned that in practice the explicit form of φ_x' is not often required; its existence is the important fact. Henceforth we indicate the reciprocal of a given associate by accenting the symbol of the latter as above.

THEOREM VI. *There is a unique* φ_x *satisfying the relation* $(\varphi, \mu)_x = \omega_x$ *when* μ_x, ω_x *are given:* $\varphi_x = (\omega, \mu')_x$.

For, from the given relation, $(\omega, \mu')_x = ((\varphi, \mu), \mu')_x = (\varphi, \mu, \mu')_x = (\varphi, (\mu, \mu'))_x = (\varphi, \eta)_x = \varphi_x$. In the same way, or as a corollary, we have the general case of such inversion:

THEOREM VII. *If* $(\varphi, \psi, \cdots, \lambda, \chi, \sigma, \cdots, \mu)_x = (\pi, \tau, \cdots, \omega)_x$, *then* $(\chi, \sigma, \cdots, \mu)_x = (\pi, \tau, \cdots, \omega, \varphi', \psi', \cdots, \lambda')_x$.

An interpretation in the numerical case of Theorem VI is as given in § 3 of the paper cited in § 4. For let the associates be single-valued functions of their arguments. Then if $\varphi_i(n) = 1$ for $n > 0$ an integer in the formula $\varphi_i = \varphi_k\varphi_i'$ stated in that paper, $\varphi_i'(n)$ is Möbius' $\mu(n)$, and the result is Dedekind's inversion, which thus appears as a very special consequence of the theorem in the numerical case.

6. *The Special Field of all Associates.* For greater clearness, before stating the next theorem, we recall the significance of the terms involved. For a rigorous discussion of the relevant logic we must refer the reader to *Principia Mathematica,* volume 1, pages 15–27, 85–88, 143–156, 260–268, 278–291. The term *real variable* is used as defined on page 18 of that work. The elements of an abstract field are real variables; theorems relating to an abstract field are assertions of propositional functions (ibid., pp. 15, 19). Our fundamental assumption in § 2 (ii) is "associates admit of unambiguous combination by formal rational operations with associates." Thus, in particular, it is assumed that in the abstract field φ_ξ ($\xi =$ any element of A, $\varphi_\xi =$ any associate of ξ), statements of identity between formal sums, differences, products, and quotients of elements (associates) in the field are propositional functions. On referring to the definitions of *extended Dirichlet product* and *reciprocal,* we see that only formal sums and products of associates are involved in the former, and only formal division by a power of ϵ, formal sums, differences and products of associates are involved in the latter. Hence statements of identity between Dirichlet products and reciprocals are propositional functions; and Theorem VIII is (*Principia Mathematica,* p. 18) an "ambiguous assertion" for every such propositional function in the abstract field of associates. These propositional functions in every case, *when Dirichlet products and reciprocals are written in full,* reduce to identities in *formal* addition ($+$), subtraction ($-$), multiplication (ab, etc.), division (a/b), when, as assumed, these operations have with respect to associates $a, b, \cdots$ their usual formal properties, $a + a - a = a$, $ab = ba$, $a(b + c) = ab + ac$, etc.

When in Theorem VIII special meanings are assigned to the associates and operations, it becomes a proposition concerning a special class of associates. An important case is that in which the associates are single-valued functions. To formal addition in this case we give the following meaning: The elements x are those of A, § 2 (i). If φ_x has a unique value

$\overline{\varphi}_x$ for each element x, and ψ_x a unique value $\overline{\psi}_x$ for each x, then the single-valued function having for each element x the value $\overline{\varphi}_x + \overline{\psi}_x$ is called the sum $\varphi_x + \psi_x$ of φ_x and ψ_x. Similarly for formal subtraction. For formal multiplication in this case, the single-valued function having for each element x the value $\overline{\varphi}_x\overline{\psi}_x$ is called the *product* $\varphi_x\psi_x$ of φ_x and ψ_x; and similarly for division. And so in each special case of Theorem VIII, the indicated *specific* operations of Dirichlet multiplication as defined in § 4 and division (extended Dedekind inversion) as in Theorem VII, may be said to have been broken down into their equivalents in terms of formal operations, to which are given special interpretations according to the special meanings assigned to the associates. Combining Theorems II, III, V, VII we have now the following powerful result.

THEOREM VIII. *All associates of the elements of an abelian group constitute a special field in which addition and subtraction are formal, multiplication is extended Dirichlet multiplication and division is extended Dedekind inversion.*

7. *Extended Cauchy Product.* Although Theorem VIII is sufficient for most purposes, it is of interest in many applications to proceed from it to an arithmetic classification of the results. This is not always feasible, but a majority of the known theorems on the arithmetic functions mentioned in the introduction, and the entire body of results of which these are but a part, can be so classified upon the following simple basis.

Restating Landau's definition (loc. cit., p. 670) of Cauchy multiplication, we call the class $[\alpha, \beta]_s$ $(s = 0, 1, \cdots)$ the *Cauchy product* of the classes α_s, β_s $(s = 0, 1, \cdots)$, when $[\alpha, \beta]_s = \sum_{r=0}^{s} \alpha_r\beta_{s-r}$. Repeating the process for $[\alpha, \beta]_s$ and γ_s $(s = 0, 1, \cdots)$, we form the class $[\alpha, \beta, \gamma]_s$ $(s = 0, 1, \cdots)$, and so on, getting finally, for any number of classes $\alpha_s, \beta_s, \cdots, \delta_s$ $(s = 0, 1, \cdots)$, the Cauchy product of all of those classes, $[\alpha, \beta, \cdots, \delta]_s$ $(s = 0, 1, \cdots)$. We call $\alpha_s, \beta_s, \cdots, \delta_s$ the factors of $[\alpha, \beta, \cdots, \delta]_s$. Clearly $[\alpha, \beta, \cdots, \delta]_s$ is invariant under permutation of some or all of its factors (this is equivalent to permuting $\alpha, \beta, \cdots, \delta$ within $[\alpha, \beta, \cdots, \delta]_s$); and if $\alpha_s + \beta_s = \gamma_s$ $(s = 0, 1, \cdots)$, then $[\gamma, \delta]_s = [\alpha, \delta]_s + [\beta, \delta]_s$. That is, this multiplication is distributive.

THEOREM IX. *Extended Cauchy multiplication is associative, commutative, and with respect to formal addition, distributive.*

The important case arithmetically is that in which the factors of a Cauchy product are single-valued functions. Let $x = x_a^\alpha x_b^\beta \cdots x_c^\gamma$ as in preceding sections. Then the following theorem is an important consequence of the definitions.

THEOREM X. *The sign* Π *of formal multiplication extending to all* $(\rho, r) = (\alpha, a), (\beta, b), \cdots, (\gamma, c)$,

$$(\varphi, \psi)_x = \Pi[\varphi(x_r), \psi(x_r)]_\rho \equiv \Pi[\sum_{s=0}^{\rho} \varphi_s(x_r)\psi_{\rho-s}(x_r)].$$

By repetition, we reach the following general result:

THEOREM XI.

$$(\varphi, \psi, \cdots, \mu)_x = \Pi[\varphi(x_r), \psi(x_r), \cdots, \mu(x_r)]_\rho.$$

From Theorems IV and X, we arrive at the inverse process, which we shall call *extended Cauchy division,* in the following theorems.

THEOREM XII. $[\varphi(x_a), \varphi'(x_a)]_a = \eta;$

THEOREM XIII. *The first of the following relations implies the second,*

$$[\varphi(x_a), \cdots, \lambda(x_a), \chi(x_a), \cdots, \mu(x_a)]_a = [\pi(x_a), \cdots, \omega(x_a)]_a,$$
$$[\chi(x_a), \cdots, \mu(x_a)] = [\pi(x_a), \cdots, \omega(x_a), \varphi'(x_a), \cdots, \lambda'(x_a)]_a.$$

Theorem XIII is analogous to Theorem VII, from which, in conjunction with Theorem X for $x \equiv x_a^\alpha$, it can easily be proved. Combining these results we have the analogue of Theorem VIII.

THEOREM XIV. *All the associates of the elements of an abelian group constitute a special field in which addition and subtraction are formal, while multiplication and division are extended Cauchy multiplication and its inverse.*

We shall call an abelian group of finite or infinite order which has a unique basis,* *an arithmetic system.* This name is chosen because such a group has the characteristic distinguishing arithmetic from algebra, that is, a unique factorization law. Arithmetic systems are abstract or special; in the former multiplication is abstract, in the latter, specific. Let us assume as a hypothesis that the class of all associates is an arithmetic system under extended Cauchy multiplication. Let us call the elements of the basis the *prime associates of the system:* a prime associate is the extended Cauchy product of no pair of prime associates each distinct from ϵ, η. From Theorem XI, we have then the following theorem.

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* Cf. Weber, loc. cit., p. 33.

THEOREM XV. *The specific arithmetic system of associates in which multiplication is extended Cauchy multiplication is also a specific arithmetic system in which multiplication is extended Dirichlet multiplication.*

8. *Connection with Formal Expansions.* With each element x_i $(i = 0, 1, \cdots)$ of an abelian group (or of an arithmetic system) let us associate an umbra z_i, and let us write

$$\overline{\varphi}(x, z) = \Pi_i[\varphi_0(x_i)z_i^0 + \varphi_1(x_i)z_i^1 + \cdots + \varphi_n(x_i)z_i^n + \cdots]$$

and similarly for $\overline{\psi}(x, z), \cdots, \overline{\mu}(x, z)$. Then the coefficients of $z_a^{\alpha} z_b^{\beta} \cdots z_c^{\gamma}$ in the formal developments of $\overline{\varphi}(x, z)$, $\overline{\varphi}(x, z)\overline{\psi}(x, z) \cdots \overline{\mu}(x, z)$, and the reciprocal of the latter, are, respectively, φ_x, $(\varphi, \psi, \cdots, \mu)_x$, $\eta/(\varphi, \psi, \cdots, \mu)_x$.

9. *Note.* We have nowhere intended to imply that the processes of this paper are more than *extensions* of Dirichlet multiplication and of Dedekind inversion. In fact it can be shown that the processes of this paper and those current in the theory of numbers are formally equivalent in the sense that either set may be inferred from the other. The forms in this paper are so stated that they provide powerful algorithms, in which all discussions of convergence are obviated by establishing identities between arithmetic functions. The theory can be generalized to associates $\varphi_{x, y, z, \ldots}$ of any number of independent classes (or abelian groups) of elements, but the generalization has no apparent interpretation in terms of ordinary numbers.

THE UNIVERSITY OF WASHINGTON.

CHARLES LEONARD BOUTON*

A MINUTE READ BEFORE THE FACULTY OF HARVARD UNIVERSITY

March 28, 1922

Charles Leonard Bouton was born in St. Louis, Missouri, April 25, 1869. His father, William Bouton, was of Huguenot descent, and the family was long established in New England. At the close of the Civil War, William Bouton settled in St. Louis, where his regiment had been disbanded. Charles's mother, Mary Rothery Conklin, was also of old American stock; her grandparents were Scotch. William Bouton was an engineer by profession. His grandfather is said to have been the projector of the Erie Railroad, and was the author of the first article on its construction. Charles was the only one of the four sons who did not follow in his father's footsteps. The home atmosphere was academic and intellectually stimulating.

Bouton received his early education in the public schools of St. Louis, and took his first degree, that of Master of Science, at Washington University in 1891. Here, he came under the instruction of a highly skilled teacher of descriptive geometry, Dr. Edmund Arthur Engler. The next two years were given to teaching in Smith Academy, St. Louis, and these were followed by a year as instructor in Washington University, part of his work being to assist Professor Henry S. Pritchett. His next, and as it turned out, his last move was to Harvard. The years '94–'95 and '95–'96 were spent in the Graduate School. He took the master's degree at the end of the first year, and at the end of the second he was awarded a Parker Fellowship for study abroad. His two years at Leipzig were most profitably spent. He chose as his master that most original geometer, Sophus Lie, then at the height of his fame. As a matter of fact, Bouton was one of the great Norwegian's last pupils, for Lie returned to Norway in 1898 and died soon after. All of Bouton's subsequent scientific work bore the clear impress of Lie's genius. His two advanced courses, which he originated soon after his return to Harvard, dealt respectively with the theory of geometrical transformations and the application of transformation groups to the solution of differential equations. The graduate students who subsequently had the good fortune to prepare for the doctorate under his care generally took up subjects connected with the theory of transformations.

After receiving the doctorate at Leipzig in 1898 Bouton returned to Harvard and began a long period of work, broken only by occasional sabbatical absence. He threw himself with the greatest zeal into his duties as a teacher. At one time or another, beside the alternating advanced courses mentioned, he taught nearly every one of the lower and middle group courses in mathematics. No pains were too great for him to spend, either on the preparation of lectures or on helping the individual student, whether a Freshman or a candidate for the doctor's degree. His characteristic quality of scientific sanity was invaluable, for it led him always to emphasize that which was permanently important, and to avoid tinsel and sham. A fine example of his didactic sense is seen in a collection of problems on the construction of Riemann's surfaces, published in volume 12 (1898) of the ANNALS OF MATHEMATICS. He was equally successful in arousing the interest of a beginner by showing him a model or a diagram or an enlightening example of a new theory, and in guiding a graduate with sure hand toward researches of permanent value and importance.

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* Professor Charles Leonard Bouton died on February 20, 1922. See this BULLETIN, vol. 28, p. 82 (Jan.–Feb., 1922).

Those qualities which made Bouton an admirable teacher were conspicuous in his other professional activities. He was an editor of the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY from 1900 to 1902, and an associate editor of the TRANSACTIONS of the same society from 1902 to 1911. His power of keen yet kindly criticism, and his unerring mathematical judgment made him an efficient referee. His advice was prized by all who knew him, his opinion was always heard with respect, and his sanity was no less remarkable than his unselfishness. All of these qualities were drawn upon in full measure in the autumn of 1918 when, almost overnight, he was called to organize the mathematical instruction of nearly a thousand men in the Students' Army Training Corps. He carried this work through with conspicuous success, and the leading teachers of mathematics in the schools of this community, who enthusiastically rallied to the support of Harvard and the nation in that crisis, found in him a helpful guide and an efficient administrator.

His home life was beautifully quiet and peaceful. In 1907 he married Mary Spencer of Baltimore, and she, with their three daughters, Elizabeth, Margaret, and Charlotte, survives him. Yet for some time before the end, long dark shadows were crossing his life. The persistent after-effects of a hurried operation for appendicitis seemed to sap his strength. Family cares and anxieties multiplied, reaching a crisis in 1918 with the death of his youngest child. His breakdown in 1921 seemed but the inevitable end toward which events had long been tending. His death deprived the university of a faithful servant, and the community of a single-minded and upright gentleman.

WILLIAM F. OSGOOD
(Signed) JULIAN L. COOLIDGE
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HARVARD UNIVERSITY.

KLEIN'S COLLECTED WORKS

Felix Klein: Gesammelte mathematische Abhandlungen. Vol. 1. *Liniengeometrie, Grundlegung der Geometrie, Zum Erlanger Programm.* Edited by R. Fricke and A. Ostrowski. Berlin, Julius Springer, 1921. xiii + 612 pages.

Felix Klein, born at Düsseldorf, April 25, 1849, was awarded the doctor's degree December 12, 1868, by the University of Bonn. The dissertation was directed and the examination conducted by Professor Lipschitz. At the golden jubilee of the doctorate, celebrated at Göttingen, December 12, 1918, plans were perfected for the publication of the numerous and varied mathematical contributions he made during the ensuing fifty years.

This was a particularly fitting thing to do. During this time no one exercised a greater influence on the development of mathematics in Germany than he. That that influence was not confined to Germany is attested by the long list of Americans who have taken the doctorate under his direction, and the very large number who have come from all parts of the world to hear his lectures or to participate in his Seminar. Notwithstanding the excellence of his personal contributions, and the inspiration of personal contact with him, probably his strongest characteristic was his skill as an organizer, and his capacity for cooperation.

Klein is essentially a geometer, and he began his work just in the zenith of the teaching of Clebsch, Kummer, Plücker and von Staudt; a worthy member of the group which included Lie, Darboux, Cremona, Clifford, Noether . . . How important are their achievements in the study of the more critical concepts of number, integral, limit, point set, etc., of to-day!

Of the three sub-titles indicated, the memoirs on line geometry occupy the first 240 pages. While the text is nowhere materially changed, a few corrections are introduced, indicated by square brackets, a considerable number of comments so marked are found in the footnotes, and each longer memoir or group of related memoirs is prefaced by biographical material written by the author, which add not only to the human interest, but aid in the understanding of the relations between various questions considered.

While at Bonn, Klein was Plücker's assistant, not only in the preparation of lecture notes on theoretical physics, but also in the writing of the *Line Geometry.** Klein intended to make physics his chief study, but acquired so much momentum in this work that he could not stop. At the time of Plücker's death the first part of the *Line Geometry* was practically all in print, but much of the second part was undeveloped. Plücker had started Klein on the problem of the quadratic complex, and the supervision of his progress was now taken over by Lipschitz. Plücker's treatment of the new idea was detailed and elementary, strikingly different from the style of Battaglini, who wrote on the quadratic complex shortly after Plücker's earlier papers had appeared.* Klein discovered that Battaglini's form was not the most general one, and presented the manuscript of a thesis in which the complex and the quadratic identity are both expressed as the sum of squares. But in the fall of 1868 Lipschitz received the proof-sheets of Weierstrass's forthcoming paper on elementary divisors, and he asked the candidate to include the various cases in his dissertation. The principles of the classification are found in the dissertation but the details of the transformation and its geometric meaning are reserved for one of Klein's own pupils later† and the numerous inaccuracies found in the interpretation have been corrected by various Italian mathematicians. As soon as his examination was completed, Klein went to Göttingen to be with Clebsch while preparing the second part of the *Line Geometry*, and remained until the end of the following summer semester. Both Clebsch and Noether were working on the problem of mapping an algebraic surface on the plane.

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* *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement.* Part I, pp. 1–226, edited by Clebsch, 1868; Part II, pp. 227–378, edited by Klein, 1869.

The following winter semester Klein spent in Berlin; there he met L. Kiepert, O. Stolz, and S. Lie. He heard no lectures, but took an active part in the Seminar conducted by Kummer and Weierstrass. Stolz had told Klein about non-euclidean geometry, and the latter soon felt that it would furnish an interpretation of Cayley's absolute measurement, which had already made a profound impression upon him. Curiously, Weierstrass emphatically rejected the whole idea, as distance was too invariant to be subjected to change of scale. In the spring, both Klein and Lie went to Paris, where they lived in adjoining rooms until the war broke out in July. Klein reported at once for duty, but was rejected on account of frail health; later he participated in relief work until he was taken ill with typhoid fever, which kept him bedfast for several weeks. Lie visited him while he was still an invalid, and together they wrote the paper on the asymptotic lines of the Kummer surface. Klein returned to Göttingen as a docent in January, 1871. He remained there until the call to Erlangen, in 1872. In the summer of 1871, Stolz also came to Göttingen.

With the exception of two titles (numbers XII, XIII) all of the papers appearing in this first part were written during these three years. These later papers were also concerned with line geometry, but the methods employed in them are essentially transcendental. In the preface it is explained how they naturally fit into the earlier category. They are *Ueber Konfigurationen, welche der Kummerschen Fläche zugleich eingeschrieben und umgeschrieben sind*, (MATHEMATISCHE ANNALEN, vol. 27 (1885)) and *Zur geometrischen Deutung des Abelschen Theorems der hyperelliptischen Integrale* (MATHEMATISCHE ANNALEN, vol. 28 (1886)). They were suggested by the dissertations of some of Klein's students, particularly of Rohn‡ and of Domsch.§ The most important contribution of Klein to line geometry is his system of coordinates, and its application to systems of quadratic complexes, to confocal systems, and to differential properties.

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* *Intorno ai sistemi di rette di secondo grado*, ATTI DELLA ACCADEMIA DI NAPOLI, vol. 3 (1866).

† A. Weiler, *Ueber die verschiedenen Gattungen der Complexe zweiten Grades*, MATHEMATISCHE ANNALEN, vol. 7 (1872).

‡ *Betrachtungen über die Kummersche Fläche und ihren Zusammenhang mit den hyperelliptischen Functionen* $p = 2$, Munich, 1878.

§ *Die Darstellung der Fläche vierter Ordnung mit Doppelkegelschnitt durch hyperelliptische Functionen*, Leipzig, 1886.

The second part of the present volume, pages 240–410, contains a number of early notes, and the long memoir of 1890, which contains a substantial harmonization of the earlier notes on measurement, the theory of groups, and the significance of transformations which leave a given configuration —space, variety, surface, curve—invariant. Although von Staudt's *Geometrie der Lage* was written in 1847, it was not noticed until 1871 that his treatment of harmonic elements, upon which the later *Beiträge* built the theory of coordinates, was essentially independent of the parallel axiom; it was Klein who made it sharply and distinctly so. The first paper is a brief summary which first appeared in the GÖTTINGER NACHRICHTEN, showing that Cayley's theory of measurement can be forcefully visualized by means of von Staudt's projective geometry. Besides these two sources the author calls attention to the help he got from the memoir of Beltrami* and particularly to the great inspiration which he found in the writings of Clifford. He states that he found the books of Salmon most useful and instructive.

In the fall of 1873 Klein attended the Bradford meeting of the British Association, and met Cayley, Sylvester, Clifford, and Ball (Sir Robert). The two long papers *Ueber die sogenannte Nicht-Euklidische Geometrie* had already appeared before the Bradford meeting was held, as the second was finished in 1872. The next paper was written after the Bradford meeting. It contains the axiom of projectivity in its present form, and also the statement that continuity can be postulated in the same manner as in metric geometry. Now follows a gap of six years, at the end of which appears the short note in which the author acknowledges the redundancy of his system of axioms necessary for the von Staudt development; it is a further analysis of the axiom of projectivity. The last paper appeared in 1890, at the end of the course on non-euclidean geometry given at Göttingen in 1889–1890. It contains a detailed account of Clifford's theory of parallels (right and left parallels), points out the difference between spherical and elliptic geometry, develops a purely projective foundation for analytic geometry, and discusses the meaning of axioms in general.

The third volume of Lie's *Theorie der Transformationengruppen* appeared in 1893. The preface contains (pp. x–xii) an admirable resumé of the growth of non-euclidean geometry and of the significance of the problem discussed in Part V, pages 393–543. It also contains several statements regarding the contributions of Klein to the development of non-euclidean geometry and of the foundations of geometry that are sadly out of keeping with the worth and the dignity of the other parts of that monumental work. In 1897 Klein was invited to report on the value of Part V of Lie's volume 3 to the Physico-Mathematical Society of the imperial University of Kasan in connection with the first award of the Lobachschevsky prize. The report covers 18 pages in the present volume. A more dignified or noble eulogy can hardly be found in critical literature than that sent by Klein to the Russian Society, proposing Lie's name. Revenge is sweet.

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* *Saggio di interpretazione della geometria non euclidea*, GIORNALE DI BATTAGLINI, vol. 6 (1868).

The third part is concerned with the Erlangen Programm, but there is no sharp division between this and the preceding one. It begins with the joint note on W-curves, written by Klein and Lie, first the summary that appeared in the COMPTES RENDUS, then the full paper that was published a little later in the ANNALEN. In this paper one sees the different points of view of the two authors, one striving to express the phenomena as discrete operations, the other to define everything in terms of differential equations. It is a perfectly natural suggestion growing out of their joint work on non-euclidean geometry. The difference in point of view is well illustrated by a little incident, now mentioned in a footnote. The W-curves are defined as those having tangents cutting the triangle of reference in three points, which, with the point of contact, make a constant cross ratio. Klein was disturbed by the indeterminate form which arises when the curve has singular points at the vertices, and proposed that the paradox be investigated. (This has been done since in connection with the integrals of differential equations of the first order, by Poincaré, Vessiot, and others.) But Lie was so filled with the general theory that such things did not interest him. His reply was: "Die Kurve weiss selbst am besten, wie sie sich in singulären Punkten zu verhalten hat."

We now come to the Programm itself. It was prepared in the midst of the other early papers already mentioned, and while the author was still saturated with his work with Plücker, with Lie, with Stolz, and he was in daily contact with Clebsch while the actual paper was being written. To those who have worked a long time in algebraic geometry, most of the ideas there developed now seem almost self-evident. The idea of the group, as expressed in terms of birational transformations, the principal subgroup (Hauptgruppe), under which a surface remains invariant, the necessary restrictions on the families of curves on the surface, the change of space element, the geometric facts concerning invariants, are now the ordinary tools of the trade. But when we recall that this was written fifty years ago, that its author was twenty-three years old, that Jordan's treatise on substitution groups had hardly been out a year, that Cayley's papers on Cremona transformations, Noether's memoir on mapping and the Brill-Noether paper on algebraic curves were not then in existence, that Lie's line-sphere transformation appeared the same summer, then we realize that it was a veritable programme, which has been closely followed during the last half century. The contributions of the Italian geometers furnish a sufficiently eloquent tribute to the comprehensiveness of the plan there outlined. If we now re-read the preface of Volume III of Lie's *Theorie der Transformationengruppen*, especially page xvii, we are supplied with a great deal of food for serious reflection.

Klein remained at Erlangen five semesters, then went to Munich Easter, 1875, thence to Leipzig, October, 1880, and finally to Göttingen in 1886, at which time through his efforts Lie was made his successor at Leipzig. The first few years at Göttingen were devoted to systematizing and completing the earlier contributions, and occasionally to take a new glance at mechanics, which as a boy had interested him more than mathematics. Besides giving a course on mechanics he prepared the treatise on the theory of the top with Sommerfeld, the first volume of which appeared in 1897. In 1901 Sir Robert Ball's *Theory of Screws* was published. The Encyklopädie was undertaken in 1894, and Klein was appointed editor of the volume on mechanics. These facts account for the next paper, a criticism of Sir Robert's book. It emphasizes what Klein had always maintained, namely, the necessity of accounting for exceptional and singular cases, to which the general theory ceases to apply. By reading the Programm and the criticism together, one sees the close connection between them. To be consistent with his own scheme, the author should have everywhere omitted the differentials of the second order, but had this been done it would lead to the exclusion of what both the author of the book and of the criticism designate as the most important part of the theory, in those cases in which the first differentials are identically zero.

The remainder of the present volume consists of much later papers; they are concerned with the connection between the theory of relativity and the projective theory, as developed in the Programm, particularly the fundamental significance of Cayley's theory of measurement. The first paper formed the closing chapter of a course on analytic projective geometry given in 1909–1910, and makes an elementary approach to the Lorentz group, showing it to be included in the affinity group in space of four dimensions. The same also contains a clear summary of the argument for the various kinds of non-euclidean geometries of fewer dimensions, and how they are related by a continuous change of the space parameter (constant of curvature). After the elementary part, entirely algebraic, there follows a short discussion of the Maxwell electro-dynamic equations, in which it is shown that they are invariant when the space coordinates are changed linearly, or the time is translated, but not under such transformations as contain both space and time coordinates in the same equation.

The next paper is a letter to Hilbert, after the appearance of Hilbert's first note on the foundations of physics. It contains an alternate derivation of Hilbert's results by means of an appropriate transformation in the sense of the Programm. Then in reply Hilbert supplies the details of the analysis, and Klein shows, by means of a system of partial differential equations that even the general theory of relativity can be treated from this same standpoint, as well as the earlier special theory. Only those consequences of the differential equations have a physical sense that are invariant under the operations of the group—a property that is inherent in the Programm for any geometry. The last two papers of the volume apply the same transformations directly to the Einstein equations in the new theory of gravitation, leading to many material simplifications. In particular, the meaning of the symbols in the multiple integration is treated at length, bringing them into clear relation with the Grassmann symbolism. The tubular or cylinder-formed universe of Einstein is contrasted with that of constant curvature of de Sitter. The general transformation theory applies directly to the latter, whereas artificial modifications must be made before it applies to the case of Einstein.

The book is printed in large clear type on good paper, and is almost completely free from typographical errors.

VIRGIL SNYDER.

SHORTER NOTICES

Lehrbuch der darstellenden Geometrie. By Dr. Georg Scheffers. Vol. I, ix + 423 pages, 404 figures. Vol. II, xiii + 439 pages, 396 figures. Berlin, Julius Springer, 1919, 1920.

This treatise on descriptive geometry contains in the main the substance of the lectures given by Professor Scheffers at the Technical School of Berlin. It presupposes nothing but "school-mathematics" on the part of the student, which, in America, is approximately equivalent to mathematical training in college algebra, trigonometry, and analytic geometry.

The courses in descriptive geometry in the universities and polytechnic schools of continental Europe are usually distinguished by high scientific standards and are given by men who are either mathematicians, like Scheffers, or men who have at least considerable mathematical knowledge beyond the calculus. The type of courses in descriptive geometry offered in most of the American engineering schools, serving a purely utilitarian purpose, may be found in the European trade-schools and secondary technical schools of various kinds, where, naturally, no effort is made to treat the subject as a science, to place connecting links with other branches of mathematics, or to pay any attention to historical developments.

From the standpoint of the scientific critic Scheffers' *Lehrbuch* is a masterly exposition of descriptive geometry. Almost on every page one may notice the enormous advantage in the treatment of the subject by the superior insight which a deeper mathematical knowledge affords. Nevertheless, Scheffers, even from the practical standpoint of the engineer, never grows pedantic, or looses himself in some individual pet schemes.

The concepts and propositions of projective geometry are developed as far as they are of unquestionable value in the treatment and rational solution of certain characteristic problems. Any one, like Scheffers, who is familiar with modern progressive views will, of course, contend that it is impossible to write a modern scientific descriptive geometry without the knowledge of projective geometry, even when taken in the larger sense of the word. The frequent lack of such knowledge on the part of teachers and writers accounts for many obsolete methods of treatment and the omission of some of the most beautiful applications of projective geometry to descriptive geometry.

The stimulating influence by Scheffers' competent selection and presentation of the subject-matter upon the reader is increased by extensive historical references and comments on geometric topics allied to descriptive and projective geometry. Throughout the treatise the author shows that he is well informed on the essential and important phases of the modern development of constructive geometric methods.

For didactic reasons, explained in the preface, the first volume starts with orthographic projection upon a single plane. The claim of modernity which the reviewer makes for Scheffers' book may, outwardly at least, appear from the fact that at the outset we find a statement about the principle of duality and its bearing upon projective processes. Further on we find applications to the construction of roof-shaped surfaces, of sun-dials, of regular polyhedrons and crystal-forms, of ellipses and of moment-ellipses in graphic statics. The second chapter, which deals with parallel-projection upon a single plane, includes sections on cavalier-perspective, shadows under parallel light-rays, affinity and its applications, on the foundations of general axonometry (including, of course, Pohlke's famous theorem), on the theory of involution and its applications to the polar properties of the circle and the ellipse. In the third chapter we find the customary orthographic projection upon two or more planes of projection, including a valuable section on conics. Whenever possible, full use is made of the advantages which the application of projective properties of conics and the principle of affinity afford in the graphic representation of geometric forms.

The second volume contains two chapters on perspective and various applications. Chapter IV, on central projection, treats of fundamental concepts, so-called restricted perspective, construction of shadows in perspective, invariance of cross ratio, involutory perspective, applications of perspective, perspective of circle and sphere, properties of conic sections and their applications, including Pascal's and Brianchon's theorems, and so-called free perspective.

Various applications and supplementary topics, such as plane curves, surface-ornaments, topographical surfaces, surfaces of revolution, helical and cycloidal curves and surfaces, ruled surfaces, interpenetrations and shadows, and finally a brief account of relief-perspective, form the contents of the concluding fifth chapter.

The level upon which Scheffers proceeds may be judged from the fact that even a discussion of Peano's surface is included, to show the student the danger of hasty generalizations. Peano's original surface has the form $z = (y^2 - 2px)(y^2 - 2qx)$ in which p and q are positive real integers. For the sake of convenient constructive treatment, Scheffers discusses the projectively equivalent surface

$$z = -\frac{1}{10}(x^2 - 5y)(x^2 - y).$$

Every plane through the z-axis cuts the surface in a quartic which has a maximum at the origin, so that one might expect a maximum for the surface at that point. Still it is possible to trace curves on the surface passing through O having a minimum at O.

The whole treatise is carefully written and is typographically faultless. It may be heartily recommended to teachers as well as to students of descriptive geometry.

ARNOLD EMCH.

Girolamo Saccheri's Euclides Vindicatus, edited and translated by George Bruce Halsted. Chicago, The Open Court Publishing Company, 1920. 30 + 246 pp.

The original title of Saccheri's now famous work is: Euclides ab omni naevo vindicatus, . . ., which appeared in Milan in 1733, and which Halsted translates as "Euclid freed of every fleck." In English *fleck* sounds rather Teutonic, and the reviewer suspects that *flaw*, or *blemish* would sound better to the American ear. It is highly commendable that the translator has arranged the Latin and English texts in parallel pages.

If Saccheri had drawn the last consequences of his keen logic, and if he had not unfortunately dragged in the then hazy notions of infinity and continuity, he would have established the first flawless non-euclidean geometry. On account of those concepts, foreign to euclidean geometry,* the latter portions of the book, according to Halsted, are comparatively unimportant. No less a mathematician than Segre says of this book: "Nevertheless the first seventy pages (apart from a few isolated phrases), up to Proposition 32 inclusive, constitute an ensemble of logic and of geometric acumen which may be called *perfect.*"

Considering the circumstance that the translation itself does add nothing new to well known facts, it must be said that the most valuable part of the book is the introduction written by Halsted, wherein he gives us interesting and valuable information on Saccheri's *Logica demonstrativa.* This remarkable work embodies a system of which the *Euclides Vindicatus,* which appeared later, may merely be called an application, and could obviously only originate in a mind of extraordinary ability. The first edition was published in Turin in 1697, the second in Turin in 1701, the third in Cologne in 1735. In an analysis of this logic Heath says: "Mill's account of the true distinction between real and nominal definitions was fully anticipated by Saccheri." According to Halsted, "In his *Logica demonstrativa* Saccheri lays down the clear distinction between what he calls *definitiones quid nominis* or *nominales,* and *definitiones quid rei* or *reales,* namely, that the former are only intended to explain the meaning that is to be attached to a given term, whereas the latter, besides declaring the meaning of a word, affirm at the same time the existence of the thing defined or, in geometry, the possibility of constructing it. The *definitio quid nominis* becomes a *definitio quid rei*" by means of a postulate, or when we come to the question whether the thing exists and it is answered affirmatively. "*Definitiones quid nominis* are in themselves quite arbitrary, and neither require nor are capable of proof; they are merely provisional, and are only intended to be turned as quickly as possible into *definitiones quid rei,*" by means of certain postulates of existence or constructions and by means of demonstrations. Vailati, thus far the only protagonist of the *Logica demonstrativa*, says that the anticipation of Mill's distinction gives Saccheri the right to an eminent place in the history of modern logic.

On the whole, Halsted's edition of Saccheri's work is very well done and is a very praiseworthy scientific undertaking. Equally praiseworthy would be an English edition of the *Logica demonstrativa,* and we would suggest to Dr. Halsted to try to secure, for that purpose, a copy of the Cologne edition, or, for that matter, of any of the previous editions, and thus render a great service to the history of science.

ARNOLD EMCH.

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* If, according to Weyl, we define as an *euclidean number* a number which is obtained from 1 by addition, subtraction, multiplication, and division, to which is adjoined the root of a positive number, then, within this system of definite content of euclidean numbers, all constructions of euclidean geometry may be carried out. It is not necessary to pour a continuous "space-broth" over it.

Géométrie Synthétique des Unicursales de Troisième Classe et de Quatrième Ordre. By E. Bally. Paris, Gauthier-Villars, 1920. 98 pp.

As its title indicates, this book develops the properties of unicursal curves of the third class and the fourth order synthetically. These curves are homographically equivalent to the tri-cuspidal hypocycloid or to the cardioid. The discussion is therefore limited to these cycloids. The ordinary cycloid is included as a limiting case.

If E and I are two concentric circles of radii R_e and R_i, respectively, $(R_e > R_i)$, there are two families of circles tangent to both whose diameters are $R_e + R_i$ and $R_e - R_i$. Consider two circles, one from each family, meeting in a point A and touching E at E_1 and E_2. If these circles roll upon E so that E_1 and E_2 move in opposite directions with velocities proportional to the radii of the rolling circles, A remains fixed in position upon each circle and describes a hypocycloid. If the circles roll upon I so that their points of contact move in the same direction with velocities proportional to the radii, A will describe an epicycloid.

The properties of the tri-cuspidal hypocycloid and of the cardioid are developed in Chapter I, starting from the foregoing mode of generation.

Chapter II deals with the tangential properties of these curves as correlative to point properties of nodal cubics.

The two following chapters deal with properties deducible from the fact that a tri-cuspidal quartic is the transform, by quadric transformation, of a conic inscribed in the triangle of its cusps.

Chapter V deals with the nodal cubic as a projection of a twisted cubic, and the tri-cuspidal quartic as a section of the developable surface formed by the tangents to the twisted cubic.

The book is interesting in that it develops the properties of these special curves in a simple way and by elementary means, but its value would be greatly enhanced to the interested reader had full references to original sources been included. For example, no reference is made to the papers of Steiner and Cremona (CRELLE, vols. 53 and 64, respectively) upon these curves. Perhaps such references will be given in the larger work on geometry which is promised in the preface, but which has not yet appeared.

L. WAYLAND DOWLING.

Éléments de Géométrie, I et II, by Alexis-Claude Clairaut. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1920. 14 + 96 + 104 pp.

This is an opportune time for the re-publication of Clairaut's *Éléments de Géométrie* that was first brought out at Paris in 1741 and again in 1753. The present-day tendencies in the teaching of beginners' geometry are in some way those set forth by Clairaut 180 years ago. Many teachers will be glad to examine this eighteenth century rival of Euclid's Elements, a text written by one of the most distinguished French mathematicians of that century. At first Clairaut brings into play chiefly the intuitive powers. He secures the interest of beginners by frequent reference to exercises in surveying. The reprint appears in two small volumes.

FLORIAN CAJORI.

NOTES

The twenty-ninth summer meeting of the American Mathematical Society will be held at the University of Rochester, Rochester, New York, on Thursday and Friday, September 7 and 8. The regular sessions will be held in the Eastman Building. Kendrick Hall will be open for the accommodation of a limited number of men, and there are several hotels conveniently located. An excursion is being planned, and the usual joint dinner will be arranged with the Mathematical Association of America, which will hold its sessions immediately before those of the Society. Opportunity will be offered to those who desire to inspect the Research Laboratory of the Eastman Kodak Company, and the works of the Bausch and Lomb Optical Company.

Rochester is on the main line of the New York Central Railroad, and may be reached by several other roads.

Titles and abstracts of papers intended for presentation at the meeting should be in the hands of the secretary by August 10 in order to appear on the printed program. Papers received after that date can be presented only as time permits. This announcement replaces the preliminary notice issued in the past. The attention of members is invited to the fact that no other preliminary notice will be issued.

The editors of the American Mathematical Monthly request us to announce that current issues of that journal have been long delayed not only on account of the general printing situation, which has affected all mathematical journals in America, but also still further on account of difficulties incident to transfer of editorial responsibility. These obstacles seem now to be overcome, and there is hope for gradual return to normal conditions.

A number of periodicals devoted wholly or in part to mathematics have recently been founded in various European countries. Among them are the following: Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, edited by Professors W. Blaschke, E. Hecke, and J. Radon. Christiaan Huygens, Internationaal Mathematisch Tijdschrift, edited by Dr. F. Schuh and published by Noordhoff, at Groningen, Holland. Revista Matematica din Timișoara, published under the auspices of the polytechnic school at Timișoara, Roumania, and devoted mainly to elementary mathematics and problems. Spisy vydávané Přírodovědeckou Fakultou Masarykovy University (memoirs published by the faculty of sciences of the Masaryk University), Brno (Brünn), Czecho-Slavia. Warszawskie Towarzystwo Politechniche, Sprawozdania i Prace, published by the Warsaw Polytechnic Society, and containing reports of meetings of that Society and memoirs on mathematical physics.

At the meeting of the American Association for the Advancement of Science at Toronto (December 27–31, 1921), Professor G. A. Miller was elected vice-president of Section A (mathematics) and chairman of that section. The University of Toronto conferred its honorary doctorate of science on Professor E. H. Moore, president of the Association and an ex-president of the American Mathematical Society, on the occasion of this meeting.

The American Astronomical Union has appointed a committee on relativity consisting of Dr. Ludwik Silberstein (chairman) and Professors Edward Kasner, A. A. Michelson, and D. C. Wilson.

The British Association for the Advancement of Science will meet at Hull, September 6–13, 1922, under the presidency of Professor C. S. Sherrington. Professor G. H. Hardy is president of Section A (mathematics and physics) and Professor A. O. Rankine recorder of that section.

The London Mathematical Society has recently elected the following officers: H. W. Richmond, president; J. E. Campbell, A. L. Dickson and W. H. Young, vice-presidents; A. E. Western, treasurer; G. H. Hardy and G. N. Watson, secretaries.

The Belgian Mathematical Society (Cercle mathématique de Belgique) was founded at Brussels, October 20, 1921. T. De Donder is president.

In June, 1921, another mathematical society, Les Amis des Nombres, was founded, also at Brussels. Its aim is "to unite those who are primarily interested in numbers, both professional mathematicians and amateurs." The secretary of this society is A. Gérardin, and its official organ is the SPHINX-ŒDIPE, of which he has been editor for many years.

The Optical Society of America has arranged to bring out an English translation of Helmholtz's *Handbuch der physiologischen Optik* as a memorial of the hundredth anniversary of the author. Contributions are solicited to defray the expense of the project, and should be sent to Adolph Lomb, Treasurer of the Optical Society of America, care of Bausch and Lomb Optical Company, Rochester, N. Y.

In connection with the celebration of the twenty-fifth anniversary of the Chicago section of the American Mathematical Society, a fund has been collected, which is to be offered for trusteeship to the Society under the name of the Eliakim Hastings Moore Fund. It is proposed that the income be used for the promotion of mathematical research by the publication of important books and memoirs, and ultimately also by the award of prizes for important contributions to mathematics.

The class of sciences of the Royal Academy of Belgium has awarded prizes to P. Montel for his memoir *Sur les familles quasinormales de fonctions holomorphes* and to L. Godeaux for his memoir *Sur les transformations rationnelles de Jonquières de l'espace.* The following subjects are announced for prizes to be awarded in 1923: *An important contribution to infinitesimal geometry,* and *A contribution to the problem of bodies in the Einstein theory.*

The Paris Academy of Sciences announces the award of the following prizes in mathematics: the Francœur prize to René Baire, for his work in the general theory of functions; the Petit d'Ormoy prize to the late Georges Humbert, for his mathematical work; the Saintour prize to Pierre Boutroux, for his work in the theory of differential equations and in the history of sciences. The Bordin prize was not awarded and the subject (in analysis situs) was withdrawn.

The Italian Society of Sciences (the XL) has awarded its prize in mathematics for 1921 to Professor O. Tedone.

The Royal Society of London has awarded its Copley medal to Sir Joseph Larmor for his work in mathematical physics, and its Hughes medal to Professor Niels Bohr, of the University of Copenhagen, for his work in theoretical physics, especially on the structure of the atom.

The gold medal of the Royal Astronomical Society has been awarded to Dr. J. H. Jeans for his researches in stellar cosmogony.

A prize for an essay in applied mathematics, the Alan Bodey prize, has been founded at Gonville and Caius College, Cambridge University. The annual value of this prize is £ 10.

The Paris Academy of Sciences has elected Professor J. Andrade a correspondent in the section of mechanics, Professor M. Brillouin a member in the section of general physics, and Professor M. d'Ocagne a member in the section of "académiciens libres."

The Accademia dei Lincei, Rome, announces the election of Professor R. Marcolongo as member in the section of pure and applied mathematics, and of Professor G. Armellini as corresponding member and Professors A. Einstein and C. de la Vallée Poussin as foreign associates in that section.

The Italian Society of Sciences (the XL) has elected Professors E. Almansi and G. Ricci to membership.

Professor Niels Bohr has been elected a member of the Royal Institution, London.

The following advanced courses are announced for the summer session of 1922:

University of Chicago: First term, June 19–July 26; second term, July 27–September 1. By Professor E. H. Moore: Fundamental number systems of analysis, first term; Limits and series, first term. By Professor H. E. Slaught: Elliptic integrals; Differential equations. By Professor G. A. Bliss: Functions of a complex variable; Calculus of variations. By Professor E. J. Wilczynski: Metric differential geometry. By Professor J. W. A. Young: Theory of equations. By Professor F. R. Moulton: Celestial mechanics. By Professor A. C. Lunn: Statistics and probability; vector analysis. By Professor R. L. Moore: Foundations of analysis situs. By Mr. H. S. Vandiver: Theory of numbers.

Professor J. Nielsen, of the Breslau Technical School, has been appointed to a professorship at the Academy of Agriculture at Copenhagen.

Dr. R. Wavre has been admitted as privat-docent at the University of Geneva.

Dr. K. Eisenmann has been appointed professor of technical mechanics at the Braunschweig Technical School.

In the faculty of sciences of the University of Paris, Professor J. Drach has been transferred from the chair of general mathematics to that of the application of analysis to geometry, and Professor E. Cotton from the chair of theoretical physics to that of general physics (as successor to the late Professor G. Lippmann).

The following appointments to professorships in other French universities are announced: Dr. Darmois (higher analysis) at the University of Nancy; Dr. Deltheil (general mathematics) at the University of Toulouse; Dr. G. Giraud (differential and integral calculus) at the University of Clermont; Dr. P. Humbert (mathematics) at the University of Montpellier; Dr. J. Pérès (rational mechanics) at the University of Montpellier; Dr. E. Turrière (rational mechanics) at the University of Aix-Marseilles.

At the University of Rome, Professor G. Bagnera, of the University of Palermo, has been appointed professor of infinitesimal analysis, Professor F. Severi, of the University of Padua, professor of algebraic analysis, and Professor F. Enriques, of the University of Bologna, professor of mathematical methodology; and Dr. Zondadari has been admitted as privatdocent in descriptive geometry.

Professor U. Cisotti, of the University of Pavia, has been appointed professor of mathematical physics at the Milan Technical Institute.

Dr. J. H. Jeans, secretary of the Royal Society of London, has been appointed Halley lecturer at Oxford University for 1922.

Mr. W. E. H. Berwick, university lecturer at Cambridge, has been appointed to a fellowship at Clare College.

Mr. H. W. Turnbull has been appointed regius professor of mathematics at the University of St. Andrews, as successor to Sir P. S. Leng, who has resigned.

Dr. J. W. Nicholson, formerly of the University of London, has been appointed to a war memorial fellowship as tutor in mathematics and physics at Balliol College, Oxford.

Dr. L. L. Dines has been promoted to a senior professorship of mathematics at the University of Saskatchewan, and granted leave of absence for the second half of the present academic year.

Mr. F. W. Winters has been appointed assistant professor of mathematics at Dalhousie College, Halifax, Nova Scotia.

Professor F. S. Nowlan, of Bowdoin College, has been appointed lecturer in mathematics at the University of Manitoba.

Dr. J. J. Nassau has been appointed assistant professor of mathematics and astronomy at the Case School of Applied Science.

Miss Elsie M. Plapp, of the University of Wisconsin, has been appointed adjunct professor of mathematics at Hollins College.

Mrs. H. B. Newson, formerly assistant professor of mathematics at Washburn College, has been appointed professor of mathematics at Eureka College.

Associate Professor Emma L. Konantz has returned to Ohio Wesleyan University after two years leave of absence spent in teaching at Peking University.

At the University of Denver, Mr. B. F. Dostal, of Oberlin College, has been appointed assistant professor of mathematics.

Mr. Ralph Beatley, of Horace Mann School and Teachers College, Columbia University, has been appointed assistant professor of education at Harvard University; his work will lie in the field of the teaching of mathematics, and he will also give one course in the department of mathematics.

Dr. F. C. Touton has been appointed lecturer in secondary education at the University of California.

Mrs. Ethelwynn R. Beckwith, formerly assistant professor of mathematics at the College for Women, Western Reserve University, has been appointed acting assistant professor of mathematics at Vassar College.

Mr. J. C. Funk has been appointed head of the department of mathematics at the Santa Maria high school and junior college.

At the U. S. Naval Academy, Mr. G. F. Abrich, Mr. R. P. Johnson, Mr. R. C. Lamb, Mr. E. S. Mayer, and Mr. J. B. Scarborough have been promoted from instructorships to assistant professorships of mathematics.

Dr. Philip Franklin has been appointed Benjamin Peirce Instructor at Harvard University for the year 1922–23.

Recent appointments as instructor in mathematics in American colleges and universities are as follows: Brown University, Mr. H. K. Cummings; Syracuse University, Mr. I. S. Carroll; University of Colorado, Dr. Claribel Kendall (returned from leave of absence); University of Michigan, Mr. J. P. Ballantine, and Mr. W. M. Coates.

The death of Professor Charles Cailler of the University of Geneva is reported.

Professor Eduard Gubler, of Zürich, died November 6, 1921, at the age of seventy-six years.

Professor H. J. Cotterill, of the Royal Naval College, Greenwich, died January 8, 1922, at the age of eighty-six years.

Professor W. Foord-Kelcey, of the Royal Military Academy, Woolwich, died January 3, 1922, at the age of sixty-seven years.

Professor C. J. Lambert, of the Royal Naval College, Greenwich, died November 11, 1921, at the age of seventy-seven years.

Rev. J. B. Lock, formerly lecturer on mathematics and tutor at Caius College, Cambridge, died September 8, 1921, at the age of seventy-two years.

Mr. George Wentworth, co-author with Professor D. E. Smith of the Wentworth and Smith series of mathematical text-books, died August 26, 1921, at the age of fifty-three years. Mr. Wentworth had been a member of the American Mathematical Society since 1910.

The death of Professor W. W. Beman, previously announced in this Bulletin (vol. 28, p. 82) occurred on January 18, 1922, not 1921 as there stated.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

Appell (P.). Eléments de la théorie des vecteurs et de la géométrie analytique. Paris, Payot, 1921. 16mo. 147 pp. Fr. 4.00

Arwin (A.). Ueber Kongruenzen von den fünften und höheren Graden nach einem Primzahl-modulus. Stockholm, 1918. 8vo. 46 pp.

Bachmann (P.). Grundlehren der neueren Zahlentheorie. 2te, verbesserte Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 16 + 252 pp.

Bianchi (L.). Lezioni di geometria analitica. Pisa, Spoerri, 1920. 8vo. 604 pp. L. 40.00

Bieberbach (L.). Lehrbuch der Funktionentheorie. Band 1: Elemente der Funktionenlehre. Leipzig, Teubner, 1921. 8vo. 6 + 314 pp.

—— See Encyklopädie.

Bonola (R.). Die nichteuklidische Geometrie. Historisch-kritische Darstellung ihrer Entwicklung. Deutsche Ausgabung von H. Liebmann. 3te Auflage. Leipzig, 1921.

Burkhardt (H.). Funktionentheoretische Vorlesungen. Band 1, Heft 1: Algebraische Analysis. 3te Auflage. Band 1, Heft 2: Einführung in die Theorie der analytischen Funktionen einer komplexen Veränderlichen. 5te Auflage. Band 2: Elliptische Funktionen. 3te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1920–1921. 8vo.

Cantor (M.). Vorlesungen über Geschichte der Mathematik. 4ter Band. Anastatischer Neudruck. Leipzig, 1921.

Cashmore (M.). Fermat's last theorem: proofs by elementary algebra. 3d edition. London, Bell, 1921. 67 pp. 2s. 6d.

Ciani (E.). Lezioni di geometria proiettiva ed analitica. 2a edizione. Pisa, Spoerri, 1919. 8vo. 622 pp. L. 40.00

Doetsch (G.). Eine neue Verallgemeinerung der Borelschen Summabilitätstheorie der divergenten Reihen. (Diss.) Göttingen, 1920.

Einstein (A.). La géométrie et l'expérience. Traduction française par M. Solovine. Paris, Gauthier-Villars, 1921. 8vo. 20 pp. Fr. 3.00

Encyklopädie der mathematischen Wissenschaften. Band II 3, Heft 4: L. Bieberbach, Neuere Untersuchungen über Funktionen von komplexen Variabeln. Leipzig, Teubner, 1921.

Filon (L. N. G.). An introduction to projective geometry. 3d edition. London, Arnold, 1921. 8 + 253 pp. 7s. 6d.

Fuhr (H.). Zur Transformationstheorie der Fuchsschen Funktionen. (Diss., Giessen.) (Mitteilungen der Mathematischen Seminars der Universität Giessen, Heft 1.) Giessen, Selbstverlag des Mathematischen Seminars, 1921.

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MATHEMATICS
DEPARTMENT

THE FOURTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The fourteenth regular meeting of the Southwestern Section of the American Mathematical Society was held at Washington University on Saturday, November 26, 1921. The attendance included the following twenty members:

Ammerman, Ashton, Dunkel, W. W. Hart, Hedrick, Ingold, Lefschetz, Luby, Lytle, McKelvey, U. G. Mitchell, Rider, Roever, Slaught, Edwin R. Smith, Eugene Stephens, Stouffer, E. A. Weeks, Westfall, J. M. Young.

The afternoon session was a joint meeting with the Missouri Section of the Mathematical Association of America. Professor Hedrick occupied the chair during the morning session and was relieved during the afternoon session by Professor Ingold, chairman of the Missouri Section.

It was voted to hold the next meeting of the Southwestern Section at the University of Kansas. The following program committee was elected: Professors Ashton (chairman), Brenke, and Stouffer (secretary).

The titles and abstracts of the papers read at this meeting follow below. The papers of Dr. Kendall and Professor Chittenden were read by title.

1. Dr. Claribel Kendall: *Certain congruences determined by a given surface.*

In this paper a surface is considered as defined by a pair of partial differential equations of the second order as is done by Professor Wilczynski in his first memoir on the projective differential geometry of curved surfaces (TRANSACTIONS OF THIS SOCIETY, vol. 8). A line l is determined by two points which are defined with respect to a local tetrahedron of reference and in terms of the parameters of the surface. Thus a line is given for every point on the surface and hence a congruence is determined. Relative to this general congruence, formulas are found (1) for the torsal curves on the surface, i.e., the curves which give the developables of the congruence, (2) for a new class of curves which are called the guide curves of the congruence, (3) for the focal points of the line l. These formulas are applied to special congruences and various relations among the lines generating these congruences are found.

2. Professor Otto Dunkel: *The direct determination of the minimum area between a curve and its caustic.*

In the WASHINGTON UNIVERSITY STUDIES, SCIENTIFIC SERIES, vol. 8, no. 2, pp. 183–194, necessary and sufficient conditions were derived for the existence of a minimum area by methods similar to the usual methods of the calculus of variations. The present treatment by Professor Dunkel is similar to the one given by him for the solution of Euler's problem (AMERICAN MATHEMATICAL MONTHLY, vol. 28, pp. 15–19) in that advantage is taken of the fact that the integrand may be reduced to a square and that the problem is solved by elementary means without the use of the general theory of the calculus of variations. The solution of this problem differs from that of Euler's problem, notably in that most of the results are obtained by reductions to complete squares.

3. Professors E. R. Hedrick, Louis Ingold, and W. D. A. Westfall: *Classification of non-analytic functions.*

In this paper Beltrami's extensions of the Cauchy-Riemann equations are made the basis for a classification of all non-analytic functions. Two functions belong to the same class if the fundamental quantities E, F, G, for the two functions are proportional. Some properties of functions belonging to the same class are then developed.

4. Professor S. Lefschetz: *A topological group of algebraic varieties.*

The group in question is the transformation group G of the $(d-1)$-dimensional cycles of a linear pencil of hypersurfaces of a d-dimensional variety. The singularities being ordinary, the operations of G acting upon a given cycle are completely defined by means of Kronecker characteristics (the algebraic number of intersections of cycles). For d even the group is of the same type as the Picard group, defined by him for $d = 2$. For d odd the fundamental substitutions are of order *two*, G being nevertheless infinite.

5. Professor S. Lefschetz: *On multiply periodic functions.*

Given a period matrix for $2p$-ply periodic meromorphic functions, we may define for the variables a fundamental domain in the form of a $2p$-dimensional generalized ring. The zeros common to $p - 1$ suitable periodic functions define a two-cycle of the ring. This cycle can be readily expressed in terms of those of a fundamental set. Several results are obtained, the most important being the Picard-Poincaré existence theorem for the matrix, and the classification of all intermediary functions in terms of a certain fundamental set.

6. Professor W. H. Roever: *An application of the theorem of Jacobi.*

In this paper Professor Roever applies the theorem of Jacobi* to the integration of the differential equations of motion of a particle referred to a rotating frame of reference, when a constant force acts parallel to the axis of rotation.

7. Professor E. B. Stouffer: *On curves on singular ruled surfaces.*

A ruled surface in a space of five dimensions is said to be singular if every set of three consecutive generators lies in a space of four dimensions. On every such singular ruled surface there is a curve along which tangents to the surface intersect three consecutive generators. Moreover, this curve is unique unless the ruled surface lies entirely in a space of three dimensions. A system of three linear homogeneous differential equations is used for the study of these forms.

8. Professor E. W. Chittenden: *On the division of a plane by a set of points.*

The following theorem is proved. Suppose K is a plane point set, S is the set of all points of the plane, while $S - K = S_1 + S_2$, where S_1 and S_2 are two mutually exclusive domains such that every point of K is a common limit point of S_1 and S_2. Then the set K is closed and connected. That K is connected was shown by J. R. Kline under the added restriction that K be connected "im kleinen" (TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), p. 452, Lemma B).

E. B. STOUFFER,
Secretary of the Section.

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*For a statement of this theorem see, for instance, Appell, *Traité de Mécanique Rationnelle*, 4th edition, vol. 1, p. 560.

THE TWENTY-EIGHTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The twenty-eighth annual meeting of the Society and the forty-eighth regular meeting of the Chicago Section were held at the University of Toronto on Wednesday and Thursday, December 28–29, 1921, in affiliation with the Convocation week meetings of the American Association for the Advancement of Science. Dormitory and eating accommodations for the mathematical group were furnished by the University of Toronto, Knox College being almost wholly given up to this purpose. Hart House was the center of a generous hospitality which filled the hours outside the scientific program with a delightful and varied entertainment.

On Thursday morning there was a joint session with Sections B and C of the American Association for the Advancement of Science and with the American Physical Society, and on Thursday afternoon a joint session with Section A and the Mathematical Association of America. At a special Convocation, the University of Toronto conferred on Professor E. H. Moore, as president of the American Association for the Advancement of Science, the degree of Doctor of Science, *honoris causa.*

The attendance included the following eighty-three members of the Society: Archibald, G. N. Armstrong, Atchison, Bacon, Beatty, Beetle, Bliss, Bradshaw, R. W. Burgess, Cairns, H. K. Cummings, L. D. Cummings, Currier, Curtiss, Dadourian, DeLury, Denton, L. L. Dines, Dostal, Dresden, Peter Field, Fields, Findlay, Finkel, Focke, W. B. Ford, J. L. Gibson, Gillespie, Glashan, Glenn, Glover, Gummer, Haskins, Hazlett, E. R. Hedrick, Howe, Huntington, Hurwitz, Ingraham, Karpinski, Kingston, Lennes, F. P. Lewis, Marshall, T. E. Mason, G. A. Miller, Norman Miller, Mirick, E. H. Moore, F. R. Moulton, Newkirk, Olson, F. W. Owens, H. B. Owens, Leigh Page, Pell, Phillips, Pitcher, Pounder, A. V.

Richardson, R. G. D. Richardson, H. L. Rietz, Robison, E. D. Roe, Roever, Sheppard, Sinclair, Slaught, C. E. Smith, Edwin R. Smith, P. F. Smith, Synge, Uhler, Veblen, Waddell, Watkeys, Webster, Wiener, Wilder, W. L. G. Williams, A. H. Wilson, E. B. Wilson, J. M. Young.

President Bliss occupied the chair at the regular meetings on Wednesday, being relieved at the morning session by Professor P. F. Smith and at the afternoon session by Professor C. N. Haskins. The joint session of Thursday morning was presided over by Professors G. W. Stewart and W. D. Harkins, Vice-Presidents of Sections B and C respectively, and that of Thursday afternoon by Professor Oswald Veblen, Vice-President of Section A.

The Council announced the election of the following sixty-two persons to membership in the Society:

Professor William Cyrus Bartol, Bucknell University;
Professor William John Bauduit, Howard University;
Mr. Benjamin Berman, University of Maryland;
Miss Florence Lucile Black, University of Kansas;
Professor Edmond Wesley Bowler, New Hampshire College;
Professor John Wentworth Clawson, Ursinus College;
Professor Robert Cameron Colwell, Geneva College;
Professor Charles Winthrop Crockett, Rensselaer Polytechnic Institute;
Mr. Herbert Kimball Cummings, Brown University;
Professor Lloyd Slote Dancey, Carroll College;
Mr. William Whitfield Elliott, Cornell University;
Professor John Thomas Erwin, George Washington University;
Professor Earl Frederick Farnau, University of Cincinnati;
Professor Forest Almos Foraker, University of Pittsburgh;
Mr. William McKinley Gafafer, Columbia University;
Mr. Harry Merrill Gehman, University of Pennsylvania;
Professor Albert Henry Steward Gillson, McGill University;
Professor J. W. Harrell, Baylor University;
Professor George Abram Harter, University of Delaware;
Miss Eugenie Caroline Hausle, Columbia University;
Professor Ruby Usher Hightower, Shorter College;
Mr. Van Buren Hinsch, Missouri School of Mines;
Mr. Frank Adin Joy, New Hampshire College;
Professor Oscar Godfrey Lawless, Talladega College;
Mr. Maurice Moses Levita, Temple University;
Mr. Thomas Richard Long, University of Rochester;
Professor Frederick Milton McGaw, Cornell College;
Professor James McGiffert, Rensselaer Polytechnic Institute;

Mrs. Martha (Macdonald) McKelvey, Iowa State College;
Professor Neil Bruce MacLean, University of Manitoba;
Miss Ida Caroline Millay, New Hampshire College;
Professor Clifford Newton Mills, Heidelberg University;
Professor Eugenie Maria Morenus, Sweet Briar College;
Dr. Hermance Mullemeister, University of Washington;
Dean Albert O'Brien, O. F. M., St. Bonaventure's College;
Mr. Frederick Allen Parker, West Virginia Collegiate Institute;
Professor Annie McKinnie Pegram, Greensboro College;
Mr. Cecil Glenn Phipps, University of Minnesota;
Professor Joseph Romeo Plante, Viator College;
Mr. Louis Gordon Pooler, Columbia University;
Mr. Abraham Press, Patent Section, Air Service, War Department;
Professor George Edwin Ramsdell, Bates College;
Dean Otis Everett Randall, Brown University;
Professor Joseph Marshall Rankin, College of Idaho;
Mr. George Emil Raynor, Princeton University;
Professor Arthur Vernon Richardson, Bishop's College;
Professor Tom Gladstone Rodgers, New Mexico Normal University;
Professor Sidney Archie Rowland, Jr., Union College;
Professor Henry Alford Ruger, Columbia University;
Professor Joseph Seidlin, Alfred University;
Mr. Leonard Shaffer, New Hampshire College;
Professor Rowland Alfred Sheets, Denison University;
Professor Wilfred H. Shirk, University of Buffalo;
Mr. George Yale Sosnow, Prudential Insurance Company;
Professor Guy Greene Speeker, Michigan Agricultural College;
Professor Mary Cass Spencer, Newcomb College, Tulane University;
Mr. Joe Ichino Tanoue, Columbia University;
Mr. Louis Weisner, Columbia University;
Professor John Jefferson Wheeler, University of Kansas;
Mr. Daniel Everett Whitford, University of Rochester;
Professor Hugh Carey Willett, University of Southern California;
Miss Ruby Willis, Wellesley College.

The invitation extended by the University of Rochester to hold the summer meeting of 1922 at that institution was accepted, and Professors Gale (chairman), Hildebrandt, Metzler, Watkeys, and the Secretary were constituted a committee on arrangements. Professor O. D. Kellogg and the Secretary were appointed as representatives of the Society on the Council of the American Association for the Advancement of Science for 1922.

The Council voted to request that the space occupied in the Bulletin by an abstract of a paper presented to the Society should be limited to the equivalent of two hundred words of text, and that display typography be avoided.

The total membership of the Society is now 1005, including 85 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 420; the number of papers read was 175. The number of members attending at least one meeting was 273. At the annual election 169 votes were cast.

The reports of the Treasurer and of the Auditors (Mr. S. A. Joffe and Professor P. H. Linehan) were received, showing a balance of $10,604.22, including the Life Membership Fund, which now amounts to $7,528.87. Sales of the Society's publications during the year amounted to $3,222.16.

The Library now contains 6,014 volumes, excluding some 500 unbound dissertations, now being catalogued.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor R. D. Carmichael, Professor D. E. Smith.
Secretary,	Professor R. G. D. Richardson.
Treasurer,	Professor W. B. Fite.
Librarian,	Professor R. C. Archibald.

Committee of Publication

Professors E. R. Hedrick, W. A. Hurwitz, J. W. Young.

Members of the Council to serve until December, 1924

Professor J. W. Alexander,
Professor Henry Blumberg,
Professor L. L. Dines,
Professor F. R. Sharpe.

At the end of the afternoon session of Wednesday, there was a business meeting of the Chicago Section. The officers elected for the next biennium are Professor A. B. Coble, Chairman; Professor Arnold Dresden, Secretary; and Professor E. W. Chittenden.

The committee on the symposium for April, 1922, reported that the symposium lecture would be given by Professor A. B. Coble on *Cremona transformations and applications to algebra, geometry, and modular functions.*

The meeting of the Society immediately preceded that of the Mathematical Association of America. A very pleasant occasion was the joint dinner, on Friday evening, of the various mathematical, physical, and astronomical associations. At the afternoon session of Wednesday it was voted to express the thanks of the Society for the hospitality of the Royal Canadian Institution and the University of Toronto, and for the entertainment furnished by the local committee of the American Association for the Advancement of Science.

On Thursday morning the program was as follows:

I. *Atomic nuclei and extra-nuclear electronic configuration,* by Professor J. C. McLennan, retiring Vice-President of Section B.

II. Symposium on quantum theory: for Section C, Dr. R. C. Tolman; for the American Mathematical Society, Professor H. B. Phillips; for the American Physical Society, Dr. Saul Dushman.

At the joint session on Thursday afternoon, the following addresses were given:

I. *A mechanical analogy in the theory of equations,* by Professor D. R. Curtiss, retiring Vice-President of Section A.

II. *The research information service of the National Research Council,* by Professor R. M. Yerkes, of the National Research Council.

III. *Subsidy funds for mathematical projects,* by Professor H. E. Slaught.

IV. *Algebraic guides to transcendental problems,* by Professor R. D. Carmichael, retiring Chairman of the Chicago Section.

In the absence of Professor Carmichael, an abstract of his paper was given by Professor Dresden; the paper appears in full in this number of the BULLETIN. Professor Slaught's paper has appeared in SCIENCE.

Titles and abstracts of the papers read at the regular sessions on Wednesday follow below. Professor Schwatt's first paper was read by Professor Pounder. The papers of Dr. Zeldin, Professors Crum, Forsyth, and Emch, Dr. Hille, Professors Lipka, MacNeish, Everett, Dodd, Wilczynski, and Glenn, Professor Schwatt's second paper and that of Dr. Wiener and Dr. Walsh were read by title.

1. Professor C. E. Wilder: *Differential geometry of an m-dimensional manifold in a euclidean space of n dimensions.*

The differential geometry of $(n-1)$-dimensional varieties in an n-dimensional space has long been developed. More recently several authors have treated the two-dimensional manifold in n dimensions.

In this paper Professor Wilder takes up the theory of the intermediate cases and generalizes certain theorems and methods of ordinary surface theory. The method used is that of the absolute calculus of Ricci combined with n-dimensional vector analysis.

2. Professor C. E. Wilder: *Differential geometry of an m-dimensional manifold in a euclidean space of n dimensions. Second paper.*

The m-dimensional manifold in n dimensions not only has extension in different directions as has the surface in ordinary space, but also it winds its tortuous way through n dimensions as does the twisted curve in three dimensions. In this paper certain facts of curve theory are generalized to an m-dimensional manifold in n dimensions.

3. Mr. M. H. Ingraham: *A modification of Peano's postulates for positive integers.*

In this paper is studied the complete independence of Peano's postulates for positive integers, and of Padoa's modification of these postulates. This study is made by means of their complete existential theory, as introduced by E. H. Moore in his New Haven Colloquium Lectures, 1906. It is proved that these postulates are not completely independent. A modification of these systems of postulates is given which is completely independent.

4. Professors L. P. Eisenhart and Oswald Veblen: *The Riemann geometry and its generalizations.*

The authors determine the conditions under which the n-dimensional geometry of a system of *paths* defined by the solution of a system of differential equations

$$\frac{d^2x^i}{ds^2} = \Gamma_{jk}{}^i \frac{dx^j}{ds}\frac{dx^k}{ds},$$

in which the Γ's are functions of $x^1, x^2, \cdots, x^n$ subject only to the condition $\Gamma_{jk}{}^i = \Gamma_{kj}{}^i$ reduces to the Riemann geometry determined by a quadratic form

$$ds^2 = g_{ij}\, dx^i\, dx^j.$$

The conditions obtained are such as to indicate that the metric introduced by Weyl as the basis of combined electromagnetic and gravitation theory is next to the Riemann metric in order of simplicity. The paper appears in the PROCEEDINGS OF THE NATIONAL ACADEMY, February, 1922.

5. Professor R. W. Burgess: *The problem of apportionment. The method of the weighted geometric mean.*

If N representatives are to be apportioned among several states of total population P, the true quota for a state of population A is $\alpha = AN/P$, ordinarily a fraction. The number of representatives actually assigned the state must be an integer, say a. The ratio $r = a/\alpha$ or α/a, whichever is greater than unity, is fundamental in the consideration of the problem. Professor Burgess suggests a method of apportionment based on the following single postulate: That apportionment is most satisfactory for which the continued product of terms of the type r^A, one term for each state, is smallest. This in effect takes the ratio for each inhabitant of each state, and uses the product of all such ratios as the measure of unfairness. If the exponents of the r were in each case unity, i.e., if the ratio were taken once for each state rather than once for each individual, this method would, in the usual case, become equivalent to those proposed by Dr. Hill and Professor Huntington. As in this new method the quantity $[\alpha/\sqrt{a(a+1)}]^A$ plays about the same rôle as does $A/\sqrt{a(a+1)}$ in Professor Huntington's method, and $A/(a+1/2)$ in Professor Willcox's method, it may be called the method of the weighted geometric mean.

6. Professor E. V. Huntington: *Necessary and sufficient conditions in the problem of apportionment.*

The present paper compares the method of major fractions and the method of equal proportions by analyzing all the known conditions or tests (including several new ones) which have been suggested in favor of either of them.*

The real question is believed to be one of simple equality between each state and every other state, here and now; and the tests based on this idea point clearly to the method of equal proportions.

Tests which attempt to compare each state directly with a norm for the whole country are found to be, in general, indeterminate or unworkable. Tests based on the idea of a total or average deviation from the norm lead to so many different methods that a choice among them appears difficult to make, especially when the averaging process is extended over unlimited time.

7. Dr. S. D. Zeldin: *Commutativity of contact transformations of mechanics.*

It is well known that with every conservative dynamical system, defined by its potential energy (a function of the generalized coordinates) and its kinetic energy (a quadratic function of the generalized velocity components), an infinitesimal contact transformation is associated whose characteristic function is of a special type. In this paper Dr. Zeldin proves that if two contact transformations associated with dynamical systems, defined as above, are independent, they cannot be commutative.

8. Professor G. A. Miller: *Substitutions commutative with every substitution of an intransitive group.*

This paper appears in full in this number of the BULLETIN.

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* For full bibliography up to the middle of 1921 (with special reference to the work of J. A. Hill) see *A new method of apportionment*, by E. V. Huntington, QUARTERLY PUBLICATION OF THE AMERICAN STATISTICAL ASSOCIATION, September, 1921. See also *Report of the Advisory Committee to the Director of the Census*, ibid., December, 1921, and a forthcoming paper by F. W. Owens in the same journal. Letters from E. V. Huntington and W. F. Willcox have appeared in the NEW YORK TIMES for Oct. 18, Oct. 23, Nov. 13, and Dec. 11, 1921. Recent papers have been presented to the American Mathematical Society by E. V. Huntington, Sept. 8, and R. W. Burgess, Dec. 28, 1921.

9. Professor G. A. Miller: *Seeming contradictions in the theory of groups.*

There are at least three distinct definitions of the term "same group" in common use in the theory of groups of finite order: (1) two substitution groups are the same if and only if one can be transformed into the other by a substitution; (2) two substitution groups or two abstract groups are the same if and only if a simple isomorphism can be established between their substitutions or operators; (3) two groups are the same if and only if one is obtained from the other by an interchange of operators. Various seeming contradictions that arise from the use of these different definitions are pointed out.

10. Professor W. A. Hurwitz: *Convergence-factors in Cesàro-summable series.*

Sufficient conditions for the theorem on convergence-factors for Cesàro-summable series have been given by C. N. Moore, Hardy, Bromwich and Chapman. Results essentially identical with part of the theorem have been proved by Schur, Carmichael and others. By means of a slight further generalization of the Silverman-Toeplitz criterion as extended by Hildebrandt and Carmichael, the author establishes necessary and sufficient conditions for the convergence-factor theorem in the case of Cesàro-summability of any order, real or complex, with positive real component.

11. Dr. W. L. Crum: *Note on the determination of the rectilinear secular trend of an ordered series of statistical relatives.*

In the analysis of certain historical series, the question arises whether, in determining the position of the line by the method of least squares, the relative deviations should not be used rather than the absolute deviations. The paper determines the line, to a first approximation, by using relative deviations, and shows that the divergence from the line as ordinarily determined is not likely to be of importance for practical applications; but there are significant theoretical consequences.

12. Professor C. H. Forsyth: *Provisions for depreciation based directly upon appraisal.*

Practically all methods of computing the depreciation charge succeed in providing a fund which will accumulate and ultimately meet the expense of replacing the property at the end of its lifetime, but no definite relation is maintained between the amount in the fund and the total amount of depreciation at any intermediate date. A method of computing the annual depreciation charge is given wherein estimates of periodic depreciation by an efficient appraiser are the foundation and accumulations of interest upon the fund are employed to reduce later charges. The scheme then insures a constant equality between the true wearing value and the theoretical wearing value with a degree of accuracy which will depend upon the reliability of the appraiser's estimates.

13. Professor Arnold Emch: *Plane algebraic curves invariant under a given quadratic Cremona transformation.*

The transformation which the author chooses in this paper has the well known simple form $\rho x_1' = x_2x_3$, $\rho x_2' = x_3x_1$, $\rho x_3' = x_1x_2$, with $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3\ (0, 0, 1)$ as the base, and $B(1, 1, 1)$, $B_1(-1, 1, 1)$, $B_2(1, -1, 1)$, $B_3(1, 1, -1)$ as the invariant points. The following theorem is proved by means of a net of invariant cubics: Let C_n be an invariant n-ic under the given transformation with ordinary multiple points of possible orders $\mu_1, \mu_2, \mu_3, \nu, \nu_1, \nu_2, \nu_3$ at the A's and B's respectively. Then $\mu_1 + \mu_2 + \mu_3 = n$, and the lines joining corresponding points of the transformation on the C_n envelope a curve $\mathfrak{C}_m$ of class $m = n - \frac{1}{2}(\nu + \nu_1 + \nu_2 + \nu_3)$. Conversely, the locus of pairs of corresponding points on the tangents of any curve $\mathfrak{C}_m$ is a certain invariant C_n. All invariant C_n's may be generated in this manner. When the $\mathfrak{C}_m$ is in a general position, then C_n is of order $n = 3m$. In case of a rational $\mathfrak{C}_m$, the C_n is, in general, hyperelliptic.

As an application Professor Emch studies hyperelliptic sextics, and rational invariant curves.

14. Professor Joseph Lipka: *Canonical systems and the general problem of dynamics.*

In this paper, Professor Lipka first derives the most general infinitesimal homogeneous contact transformations in $2n$ variables, $x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n$, connected by a relation $H(x_1, \cdots, x_n, p_1, \cdots, p_n) =$ constant, which leave the Pfaffian $\Sigma_i p_i dx_i$ invariant. Interpreting this result properly in a metric space defined by $ds^2 = \Sigma_{ik} a_{ik} dx_i dx_k$, the equations of the transformation reduce to the canonical equations of the dynamical trajectories in a conservative field of force in a curved space of n dimensions. We thus get a very elegant and very brief proof of the characteristic property of orthogonality of a system of dynamical trajectories (or of any natural family of curves)—including both the direct theorem of Lipschitz* and its converse as given by the writer.†

15. Professor H. F. MacNeish: *Euler squares.*

Euler proposed the following problem called "the problem of the 36 officers": Six officers of different rank are chosen from each of six different regiments. It is required to arrange them in a solid square so that no two officers of the same rank or of the same regiment shall be in the same row or the same column. This is equivalent to the problem of arranging in a square array the 36 distinct two-digit numbers with digits 1, 2, 3, $\cdots$, 6 so that there are no repetitions in the columns or in the rows either in the units' digits or in the tens' digits. This array is called an Euler square of order 6, degree 2 and index 6, 2. Euler stated without proof that the solution is impossible for order $n \equiv 2 \pmod{4}$. By associating the Euler square with a geometric configuration, the author gives a proof by analysis situs. Methods are given for constructing Euler squares by means of substitution groups and by combining those of lower order. Euler squares are applied to the problem of making schedules for tournaments.

16. Professor H. S. Everett: *The expression of general forms as determinants whose elements are forms.* Preliminary report.

The author obtains the following theorems: (1) If a general form of degree ρr in $n \geqq 2$ variables has more than $[n(n+1) \cdots (n+\rho-1)/\rho! - 2]r^2 + 2$ terms, it cannot be expressed as a determinant of order $r \geqq 2$ whose elements are forms of degree $\rho \geqq 2$ in the n variables. (2) A general form of degree ρr in $n \geqq 2$ variables cannot be so expressed if $n > 3$. These theorems are extensions of similar theorems in a paper by L. E. Dickson (TRANSACTIONS OF THIS SOCIETY, April, 1921), in which general forms expressible as a determinant with *linear* elements are completely determined.

17. Professor E. L. Dodd: *The arithmetic mean of the least and greatest of n measurements.*

Let $P(x)$ be the probability that the error of a measurement will be algebraically less than x, with $P(b) = 1$, b finite or

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* JOURNAL FÜR MATHEMATIK, vol. 74 (1869).

† PROCEEDINGS OF THE AMERICAN ACADEMY, vol. 55, No. 7, June, 1920.

$+\infty$; and let $P(x)$ have a continuous derivative $\phi(x)$. Let $\psi(z)$ be the probability density of the error z of the arithmetic mean of the least and greatest of n measurements; i.e., $\psi(z)dz$ is the probability that this error lies between z and $z + dz$. Then

$$\psi(z) = 2n(n-1)\int_z^b [P(x) - P(2z - x)]^{n-2}\phi(x)\phi(2z - x)dx.$$

Suppose that the error curve $y = \phi(x)$ for the individual measurements meets the X-axis perpendicularly at $\pm b$, like certain Pearson curves, and that the slope of this curve is numerically greater than any given G in some vicinity of $\pm b$. The curve $y = \phi(x)$, however, need not be symmetrical. Then, asymptotically for large n, the probability that the arithmetic mean of the least and greatest measurement will differ from the true value by less than e is greater than the probability that the arithmetic mean of all the measurements will so differ. Again, if $\phi(x) = c$, a constant in $(-1/2c, 1/2c)$, the arithmetic mean of the least and greatest measurement is better than the arithmetic mean of all the measurements.

18. Dr. Einar Hille: *Convex distribution of the zeros of Sturm-Liouville functions.*

Dr. Hille considers the zeros of a solution of a differential equation $w'' + G(z)w = 0$, in a convex region B in which $G(z)$ is analytic and $\vartheta_2 \leqq \arg G(z) \leqq \vartheta_1$, $(\vartheta_1 - \vartheta_2 < 2\pi)$. It is shown that all the zeros of $w(z)w'(z)$ in B, where $w(z)$ is a solution, can be joined by a polygonal line p_0, composed of oriented line-segments, each of which forms an angle θ_P with the positive real axis subject to the condition $-\vartheta_1/2 < \theta_P < -\vartheta_2/2$. Furthermore, let the zeros $a_1, a_2, \cdots, a_n$ be numbered in the order in which they occur on p_0. If $\arg G(z)$ is never increasing along a certain set of polygonal lines of the same type as p_0, then $\arg(a_{\nu+1} - a_\nu)$ is a never-decreasing set of angles. In this case the points (a_ν) form the vertices of a convex polygon. Finally a *zero-free* region is determined for a given particular solution; the form of this region depends only on the argument of $G(z)$.

19. Professor D. R. Curtiss: *On Kellogg's diophantine problem.*

Professor Kellogg has proposed the problem of determining the maximum x that can occur in a solution in positive integers of the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

In a recent paper (AMERICAN MATHEMATICAL MONTHLY, vol. 28, p. 300) he gives reasons for believing that the maximum x is u_n, where $u_1 = 1$, and $u_{k+1} = u_k(u_k + 1)$. Professor Curtiss proves this by solving the above equation for x_n and showing that the maximum value of the resulting expression for integral values of $x_1, x_2, \cdots, x_{n-1}$ which make it positive and finite but not necessarily integral, is u_n as defined above.

20. Dr. J. S. C. Glashan: *The isodyadic quintic equation.*

The object of this paper is to enable one to write down any number of isodyadic quintics independently of any knowledge of their roots. This is done by equating to zero the constant term of the diacrinic sextic given in the author's paper, *On the determination and solution of the metacyclic quintic equations with rational coefficients*, AMERICAN JOURNAL, vol. 23, p. 49.

21. Dr. J. S. C. Glashan: *On the isodyadic septimic equations.*

The object of this paper is to determine whether any given septimic equation is isodyadic, and, if it be so, to solve it. The general discussion is followed by application to the investigation and solution of four equations with numerically assigned coefficients. Following these examples there are given the solutions of the adyadic septimics and De Moivre monodyadic septimic.

22. Professor C. F. Gummer: *Criteria for relative root distributions.*

In a paper presented in September, 1920, the author developed a rational method to determine the mutual arrangement of the real roots of a system of real polynomials. In the case where $g(x)$ has roots $\beta_1, \beta_2, \cdots, \beta_q$, all real and distinct, and arranged in descending order, it was found, under reasonable restrictions, that the number of roots of $f(x)$ between β_r and β_{r+1} could be evaluated as the coefficient of t^r in a generating function of the form $\sum_{s=1}^{q} P_s(1+t)^{q-s}(1-t)^s$. The constants P_s were there determined by means of the Sturmian functions of $g(x)$. In the present paper it is shown that P_s is the signature of a certain quadratic form whose coefficients are symmetric functions of the roots of each of the polynomials f and g. No use is made of Sturmian functions. When g has k pairs of imaginary roots, the same rule will determine the proper generating function after removal of a certain extraneous factor. A modification of this method is also given which, while depending on a less simple quadratic form, is more easily connected with the coefficients of f and g.

23. Professor Samuel Beatty: *The algebraic theory of algebraic functions.*

The general aim kept in view in the paper is to attain the simplicity and flexibility of treatment implied in deriving properties relative to a given basis from those relative to certain appropriate bases, the study of which presents less difficulty. Use is made of order-numbers of a certain type of adjointness relative to a given value of the variable. A lower bound is obtained for the number of linearly independent reduced forms of rational functions built on certain bases relative to a given value of the variable and containing none but negative powers of the element. Upper and lower bounds are obtained for the number of linearly independent reduced forms of rational functions built on a basis,—in the latter case, a basis of a certain type. The proof of the complementary theorem is effected by noting that were it to fail in any given case certain of the numbers obtained as lower bounds would not be such. The complementary theorem is used to obtain the number of conditions applicable to the reduced form of a rational function of a certain general type to build it on a given basis relative to a given value of the variable.

24. Professor I. R. Pounder: *An algebraic proof of the existence of the branches of an algebraic function.*

Under the assumption that the repeated application of Newton's process fails to separate the branches of an algebraic function y, subject to the relation $f(x, y) = 0$, this paper shows that there exists an expression $P(x^{1/\lambda}) + x^{t/\lambda}y'$, where P indicates a polynomial, which when substituted for y in $f(x, y)$ and $f_y(x, y)$ makes both the resulting expressions divisible by an arbitrarily high power of x. This being impossible when $f(x, y)$ contains no repeated factor of the same type as itself, the validity of Newton's process is established.

25. Professor E. H. Moore: *On the determinant of an hermitian matrix of quaternionic elements.*

Consider a square array or matrix $K_n \equiv ||a_{fg}||$ of n^2

quaternions a_{fg}, $(f, g = 1, \cdots, n)$. The matrix K_n is hermitian in case $\bar{a}_{fg} = a_{gf}$. Here for a quaternion

$$q \equiv c_0 + c_1 i + c_2 j + c_3 k, \quad (c_0, \cdots, c_3 \text{ real numbers})$$

$\bar{q}$ denotes the conjugate quaternion. In view of the non-commutativity of multiplication of quaternions there is probably no very useful definition of the determinant of an unconditioned quaternionic matrix K_n. However, for an hermitian quaternionic matrix K_n a useful determinant, det. K_n, is definable as follows, by suitable determination of the order of the n factors of the $n!$ terms t of the formal classical det. K_n. Let the factors of a term t be arranged in closed cycles $(f_1 f_2 \cdots f_k) \equiv a_{f_1 f_2} a_{f_2 f_3} \cdots a_{f_k f_1}$ of order k $(1 \leqq k \leqq n)$ with distinct indices $f_1, \cdots, f_k$ of $1, \cdots, n$, of which the leading index f_1 is the smallest. Let the cycles of the term t be arranged according to increasing orders k, those of the same order k being arranged according to increasing initial indices f_1. Then the quaternionic det. K_n is the formal det. K_n with this arrangement of the factors of each term. As an application of this definition, consider n m-partite quaternions or m-ary quaternionic vectors $\xi_g \equiv (b_{1g}, \cdots, b_{mg})$ $(g = 1, \cdots, n)$ and set $a_{fg} \equiv \bar{b}_{1f} b_{1g} + \cdots + \bar{b}_{mf} b_{mg}$ $(f, g = 1, \cdots, n)$. Then the matrix $K_n \equiv ||a_{fg}||$ is hermitian, and the vanishing of det. K_n is a necessary and sufficient condition that the vectors $\xi_1, \cdots, \xi_n$ be linearly dependent on the right, that is, that $a_1, \cdots, a_n$ not all 0 exist such that $\xi_1 a_1 + \cdots + \xi_n a_n = 0$.

26. Professor E. J. Wilczynski: *Some properties of the surfaces which represent the real and imaginary components of a function of a complex variable.*

The surfaces obtained by plotting separately the real and imaginary components, $u(x, y)$ and $v(x, y)$, of a function $w = u + iv$ of a complex variable $z = x + iy$, possess notable geometric properties which seem to have escaped notice so far. Professor Wilczynski calls attention to some of these: they are connected with the (imaginary) conjugate nets corresponding, on the u-surface and the v-surface, to the null-lines of the xy-plane, the (real) associate conjugate nets, and the (real) asymptotic nets of the two surfaces, where the term associate conjugate net is used in the sense of Green; i.e., at any surface point the tangents of two associate conjugate nets divide each other harmonically. The asymptotic net of the u-surface projects into an orthogonal isothermal net in the xy-plane which is at the same time the projection of the above-mentioned real conjugate net of the v-surface, and similarly for the asymptotic net of the v-surface. The two orthogonal nets obtained in this way in the xy-plane are so related that each bisects the angles determined by the other.

27. Professor O. E. Glenn: *Note on differential invariants.*

This note relates to differential parameters for a quaternary differential form, with the condition

$$dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 = dx_1'^2 + dx_2'^2 + dx_3'^2 - dx_4'^2,$$

and the case where the characteristic equation determining a set of covariant differentials is a reciprocal equation.

28. Professor Louise D. Cummings: *Hesse's associated points and the Weddle surface.*

The theory of eight associated points from the geometric standpoint is already classic. The purpose of the present paper is to define a point by a symmetric covariant of seven other points, to establish from this equation its geometric properties, and eventually to show the point to be identical with Hesse's eighth point.

29. Professor I. J. Schwatt: *Some of the principles of the operation with series applied to a partial fraction problem.*

By means of some of the principles of the operation with series Professor Schwatt has obtained the partial fractions of

$$\left(\sum_{k=0}^{k=n} m_k x^{n-k}\right)/(x + a)^p,\ n \geqq p,$$

and

$$\left(\sum_{k=0}^{k=n} m_k x^{n-k}\right)/(x^2 + ax + b)^p,\ n \geqq 2p.$$

The coefficients of the first are expressed as single summations and those of the second as double summations. The literature on the subject shows that heretofore the coefficients have been expressed as determinants.

30. Professor I. J. Schwatt: *Expansion of powers of infinite series.*

If the series to be expanded to the pth power is not summable, the coefficient of the general term of the expansion is the product of $p - 1$ summations. If the series is summable the work is reduced to the evaluation of the general derivative of a function of a function for the vanishing primary variable. The methods employed are believed to be new.

31. Professor Olive C. Hazlett: *A symbolic theory of formal modular invariants.*

At the meeting in connection with the Chicago Colloquium, Professor Hazlett introduced a symbolic notation for the formal modular invariants of a binary form f of degree m with respect to the general Galois field $GF[p^n]$ of order p^n. To this end, the form f was expressed as the product of m symbolically distinct factors $\alpha_x = \alpha_1 x_1 + \alpha_2 x_2$, $\beta_x = \beta_1 x_1 + \beta_2 x_2$, $\cdots$. It was there shown that all formal modular invariants of f are expressible as polynomials in a finite number of symbolic expressions which behave like invariants. These invariant symbols include determinantal symbols of the type

$$(\alpha\beta) = \alpha_1\beta_2 - \alpha_2\beta_1,$$

which arise in the theory of ordinary invariants of f, and more complicated ones which are peculiar to the theory of modular invariants. In the present paper, it is shown that, if a formal modular invariant I is isobaric, it is necessarily congruent (modulo p) to a polynomial in determinantal symbols of type $(\alpha\beta)$ and hence is an ordinary invariant of f, though not necessarily a rational one. In case I is not isobaric, then it is congruent (modulo p) to an ordinary invariant (rational or irrational) of f and certain related forms. Similar theorems hold for formal modular covariants of f.

32. Dr. Norbert Wiener and Dr. J. L. Walsh: *The equivalence of expansions in orthogonal functions.*

The authors develop necessary and sufficient conditions for the identity of the convergence properties of different expansions in orthogonal functions. They apply these notions to various related problems. The chief theorem is that two closed sets of normal orthogonal functions $\{\varphi_n\}$ and $\{\psi_n\}$ have identical convergence properties when and only when the φ-series for the ψ's converge uniformly, while

$$\int_0^1 \{\sum_1^n [\varphi_n(x)\varphi_n(y) - \psi_n(x)\psi_n(y)]\}^2 dy$$

is uniformly bounded in n and x.

ARNOLD DRESDEN, R. G. D. RICHARDSON,
Secretary of Chicago Section. *Secretary.*

THE MOMENT OF INERTIA IN THE PROBLEM OF N BODIES*

BY PROFESSOR W. D. MACMILLAN

The differential equations of the problem of n bodies are

$$m_i x_i'' = \frac{\partial V}{\partial x_i}, \ m_i y_i'' = \frac{\partial V}{\partial y_i}, m_i z_i'' = \frac{\partial V}{\partial z_i}, (i = 1, \cdots, n)$$

in which the potential function is

$$V = \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k^2 \frac{m_i m_j}{r_{ij}}, \ (j \neq i).$$

The kinetic energy of the system is

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i(x_i'^2 + y_i'^2 + z_i'^2).$$

The moment of inertia of the n bodies considered as point masses with respect to the origin is

$$I = \sum_{i=1}^{n} m_i(x_i^2 + y_i^2 + z_i^2).$$

On differentiating this expression for the moment of inertia twice with respect to the time, we find

$$\tfrac{1}{2}I'' = \sum_{i=1}^{n} m_i(x_i x_i'' + y_i y_i'' + z_i z_i'') + \sum_{i=1}^{n} m_i(x_i'^2 + y_i'^2 + z_i'^2).$$

The last term of this expression is twice the kinetic energy of the system, $2T$. The first term of the right member, in virtue of the differential equations of motion, may be written in the form

$$\sum_{i=1}^{n}\left(x_i\frac{\partial V}{\partial x_i} + y_i\frac{\partial V}{\partial y_i} + z_i\frac{\partial V}{\partial z_i}\right).$$

Since V is a homogeneous function of degree -1 of the quantities x_i, y_i, z_i, we have

$$\sum_{i=1}^{n}\left(x_i\frac{\partial V}{\partial x_i} + y_i\frac{\partial V}{\partial y_i} + z_i\frac{\partial V}{\partial z_i}\right) = -V.$$

Consequently the differential equation for the moment of inertia may be written in the form

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* Presented to the Society, April 10, 1920.

$$\tfrac{1}{2}I'' = -V + 2T.$$

Since $T - V = \mathcal{E}$ is the total energy of the system, and is therefore constant, we can eliminate the kinetic energy from the preceding equation, and we obtain

$$\tfrac{1}{2}I'' = V + 2\mathcal{E}.$$

This equation was derived by Jacobi and discussed by him in considerable detail in his *Vorlesungen über Dynamik.*

Since the moment of inertia is homogeneous of degree $+2$ with respect to the quantities x_i, y_i, and z_i, and since V is homogeneous of degree -1, it follows that the product

$$VI^{1/2} = C$$

is a homogeneous function of degree zero, which in certain cases is an actual constant with respect to the time. In any event, C is a function which is always positive and which depends only upon the masses and their relative distributions, if we choose the center of mass as the origin, and not at all upon the size of the configuration. Eliminating V from the differential equation by the introduction of the function C, we have

$$I'' = 2CI^{-1/2} + 4\mathcal{E}.$$

As Jacobi pointed out, this equation shows that if $\mathcal{E}$ is a positive constant at least one of the bodies must recede to infinity, and therefore if the system is to be a permanent one, $\mathcal{E}$ must be negative. To exhibit this fact, we shall take $\mathcal{E} = -E$, and we shall suppose that we are dealing with a permanent system, so that E is positive. Then

$$I'' = 2CI^{-1/2} - 4E.$$

If C is a constant, this equation can be integrated. The first integral is

$$I^{1/2} = \frac{2}{E}[C_1^2 - (C - 2EI^{1/2})^2],$$

the constant of integration being taken to be $2[C_1^2 - C^2]E^{-1}$. Evidently the new constant C_1^2 cannot be negative. If it is zero, $I^{1/2}$ has the constant value $I^{1/2} = C/(2E)$.

The second integral is

$$\frac{C}{(2E)^{3/2}} \sin^{-1}\left(\frac{2EI^{1/2} - C}{C_1}\right)$$
$$- \frac{1}{(2E)^{3/2}} \sqrt{C_1^2 - (C - 2EI^{1/2})^2} = t - t_0,$$

which shows that $I^{1/2}$ is a periodic function of t with the period

$$P = \frac{2\pi C}{(2E)^{3/2}},$$

and that it oscillates about the mean value $C/(2E)$, the amplitude of the oscillation having the value C_1. It is worthy of note that the period of the oscillation is independent of the amplitude. It depends only upon the energy of the system and upon the nature of the configuration.*

In the problem of two bodies, the function C is necessarily a constant and has the value $(m_1m_2)^{3/2}/\sqrt{m_1 + m_2}$. In the problem of three bodies it is a constant for the equilateral triangular solution and for the straight line solution of Lagrange. It is constant for Longley's parallelogram solution of the problem of four bodies, and for Moulton's straight line solution of the problem of n bodies. In short it is constant for every solution of the problem of n bodies for which a definite geometric configuration is preserved throughout the motion, i.e., one in which the ratios of the mutual distances are constants.

This condition is sufficient, but there is nothing to indicate that it is necessary. In the globular star clusters we have a

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* If we replace $I^{1/2}$ in this integral by the radius of gyration, ρ, by means of the relation $I^{1/2} = \rho\sqrt{M}$ where $M = \Sigma m_i$; and make the further substitutions

$$\rho = \frac{C - C_1 \cos\varphi}{2E\sqrt{M}}, \qquad \frac{(2E)^{3/2}}{C}(t - t_0) = \tau - \pi/2, \qquad \frac{C_1}{C} = e,$$

it reduces to Kepler's Equation $\varphi - e \sin\varphi = \tau$, with

$$\rho = \frac{C}{2E\sqrt{M}}(1 - e\cos\varphi).$$

These equations are familiar in the problem of two bodies.

type of organization in which a steady state has apparently been reached. In such an aggregation, the function C must be very nearly constant over extraordinarily long intervals of time. If our galaxy is such a star cluster, it is oblate rather than globular. Assuming that it is composed of 1.5×10^9 stars such as our sun with an equatorial radius of 2000 parsecs (6,600 light years) and a polar radius of 600 parsecs, and that the mean velocity of the stars is 25 kilometers per second, the period of oscillation (if any exists) is 25,600,000 years and the period of a circular orbit about its equator is 74,300,000 years. If we assume a stellar density only one-fifth as great and a stellar velocity of 40 kilometers per second instead of 25 kilometers per second, the period of oscillation is 97,000,000 years, and the period of the circular orbit about the equator is 166,000,000 years. It should be added, however, that even though the function C is a constant (or nearly constant) for the galaxy, it is not necessary that there should be any oscillation. The moment of inertia may be a constant.

THE UNIVERSITY OF CHICAGO.

SUBSTITUTIONS COMMUTATIVE WITH EVERY SUBSTITUTION OF AN INTRANSITIVE GROUP*

BY G. A. MILLER

It is well known how to find all the substitutions which are separately commutative with every substitution of a transitive group G of degree n and which involve no letter except such as are found in G. If G_1 is composed of all the substitutions of G which omit a given letter of G, and if the degree of G_1 is $n - \alpha$, then all the substitutions on the letters of G which are separately commutative with every substitution of G constitute a subgroup K of order α, and all the substitutions of K besides the identity are regular and of degree n.

When G is intransitive, the results are not quite so elegant, but as they are often useful it seems desirable to state them

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* Presented to the Society, Dec. 28, 1921.

explicitly. It is evident that the direct product of the groups composed separately of the substitutions on the letters of the various transitive constituents of G which are commutative with all the substitutions of these constituents is a group K_0 composed of substitutions which are separately commutative with every substitution of G. A necessary and sufficient condition that K_0 include all the substitutions on the letters of G which are commutative with every substitution of G, is that G have no two transitive constituents which are simply isomorphic with each other and in which the subgroups composed of all the substitutions which omit a letter correspond to such sub-groups. This results directly from the following theorem:

THEOREM. *A necessary and sufficient condition that there exist a substitution which interchanges h systems of intransitivity and is commutative with every substitution of the intransitive group involving these systems, is that there be a simple isomorphism between the h transitive groups which constitute these systems, such that a subgroup composed of all the substitutions which omit a letter of one of these h transitive groups corresponds to such a subgroup of the other.*

This theorem results almost directly from the fact that every automorphism of a transitive group in which a subgroup composed of all the substitutions which omit a given letter corresponds to such a subgroup, can be obtained by transforming the transitive group by substitutions involving only the letters of this group. In fact, it is evident that we can establish an automorphism of a transitive group in such a way that a substitution t on the letters of the two transitive constituents thus obtained interchanges these constituents and is commutative with every substitution of the intransitive group formed by them. By transforming one of these constituents by a substitution on its own letters which transforms it into itself the isomorphism mentioned in the theorem is obtained, and t is transformed thereby into a substitution which is commutative with every substitution of the intransitive group obtained in this way.

If G involves h transitive constituents which can be transformed transitively by means of substitutions which are commutative with every substitution of the group formed by these constituents, then these constituents can be transformed into each other according to the symmetric group of degree h. The total number of such substitutions on the letters of the h constituents can therefore be obtained by extending the part of the direct product K_0 which relates to these constituents by means of the substitutions of this symmetric group of degree h. As a very special case of this theorem we have that all the substitutions which are commutative with every substitution of the intransitive group obtained by establishing a simple isomorphism between n symmetric groups of degree $n > 2$ and $n \neq 6$ constitute a group which is a conjugate of this intransitive group.*

From the preceding developments it results that all the substitutions which are commutative with every substitution of an intransitive group can be found by first constructing K_0 and then extending this K_0 successively by means of the symmetric groups on h transitive constituents if any exist, where h is always taken as large as possible. If one such h is not equal to the number of transitive constituents involved in G, then the group composed of all the substitutions which are commutative with every substitution of G is a direct product of the groups relating to the constituents which are transitively connected. While the order of each of the constituents of G which can be transitively connected must be the same, those which can not be thus connected may evidently have different orders.

THE UNIVERSITY OF ILLINOIS

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* Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 41. It may be noted that the condition $n \neq 6$ should have been added there since the symmetric group of degree 6 admits outer isomorphisms.

SUMMABLE INFINITE DETERMINANTS*

BY WILLIAM L. HART

1. *Introduction.* It has been customary to define the value of the infinite determinant

$$D = |a_{ij}|_{i,\,j=1,\,2,\,\cdots} \tag{1}$$

by the equation

$$D = \lim_{n\to\infty} D_n \qquad (D_n = |a_{ij}|_{i,\,j=1,\,2,\,\cdots,\,n}), \tag{2}$$

in case the limit exists. Supposing that the a_{ij} are any complex quantities, we may state

Definition. *The determinant* (1) *is summable to the value D in case the sequence* (D_n) *is summable to D.*

In this definition the term *summable* may be understood in any one of the various senses which have been given† to that term. It is easily seen that if D is summable, its value remains unchanged if we interchange its rows and columns. The sign of D is changed if two contiguous rows or columns are interchanged. If all the elements of a row, or column, are multiplied by a constant K, the new determinant obtained has the value KD.

It is the purpose of this note to present certain types of infinite determinants which are summable in the sense of the Cesàro first mean and, in the future, the unqualified word *summable* will be applied to determinants of this variety. In § 2, there will be given a proof of the summability and a discussion of certain properties of a class of determinants analogous to the Von Koch normal‡ infinite determinants. In § 3 the summability of a somewhat different type of determinants will be established.

2. *A Certain Class of Determinants.* A simple example of an infinite determinant which is summable to the value 1/2 is

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* Presented to the Society at Chicago, March 26, 1921.

† E. Borel, *Les Séries divergents*, p. 87; Hurwitz, Report on divergent series, this Bulletin, vol. 28, 1922, pp. 17–36.

‡ Cf. Kowalewski, *Determinantentheorie*, p. 372.

$$D = \begin{vmatrix} 0 & 1 & & & & \\ -1 & \pm 1 & & & & \\ & & 0 & 1 & & \\ & & -1 & \pm 1 & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & \cdot \end{vmatrix}, \tag{3}$$

where all elements are zero except for those in the main diagonal minors of order 2, which are displayed, and where the ± signs in the main diagonal may be selected at will.

For (3), we find that $D_{2k} = 1$ and $D_{2k-1} = 0$ for $k = 1, 2, \cdots$. The determinant (3) would remain summable to the value 1/2 if arbitrary elements b_{ij} were substituted for the zero elements situated above the elements $+1$, in the columns of even index, and to the left of the elements -1, in the rows of even index.

Let the arbitrary quantities b_{ij} of the preceding paragraph be chosen as either $+1$, -1, or 0, and, after this selection, let the element of (3) in row i and column j be denoted by e_{ij}.

THEOREM I. *In* (1) *suppose that* $a_{ij} = e_{ij} + c_{ij}$, *where* $\Sigma_{ij=1}^{\infty}|c_{ij}|$ *converges. Then the determinant* D *of* (1) *is summable and*

$$2D = D_2 + (D_4 - D_2) + \cdots + (D_{2h} - D_{2h-2}) + \cdots. \tag{4}$$

First let us consider certain properties of the infinite matrix

$$A = (c_{ij})_{ij=1,\,2,\,\cdots}.$$

By a minor of A of order p we shall mean a determinant of order p formed by those elements of a certain set of p rows of A which are located in a certain set of p columns. Let I represent the sum of the absolute values of all terms which enter in the expansions of all minors, of all orders, of the matrix A. Then it is well known that I converges.*

In order to establish the theorem, we note that the determinants D_n in the present case have the following properties when expanded in terms of the quantities c_{ij}:

(a) For every n, D_n is equal to the sum of certain terms from

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* Cf. Kowalewski, loc. cit., p. 374.

I, with positive or negative signs attached, plus 1, if n is even, and plus zero, if n is an odd integer.

(b) For every value of $h > 1$, D_{2h} contains all terms of D_{2h-2}, and, moreover, the new terms occurring in D_{2h} do not occur in any D_{2k} if $k < h$.

(c) For every value of h, D_{2h-1} consists of terms which are not found, even with a change of sign, in any other determinant D_{2k-1} $(k \neq h)$.

The property (a) results from the fact that the elements e_{ij} are ± 1 or 0. One verifies (b) by expanding D_{2h} according to the elements

$$c_{1,\,2h-1},\ c_{2,\,2h-1},\ \cdots,\ c_{2h-1,\,2h-1},\ c_{2h,\,2h-1} - 1, \tag{5}$$

in its $(2h - 1)$-th column. We establish (c) by expanding D_{2h-1} according to the elements of its last column, which consists of all the quantities (5) except the last.

To establish the summability of D, we must show the existence of $\lim_{n\to\infty} S_n$ where

$$S_n = \frac{D_1 + \cdots + D_n}{n}. \tag{6}$$

When $n = 2k$ let us write

$$S_{2k} = \frac{R_{2k}}{2k} + \frac{T_{2k}}{2k},$$

where

$$R_{2k} = D_1 + D_3 + \cdots + D_{2k-1}$$

and

$$T_{2k} = D_2 + D_4 + \cdots + D_{2k}.$$

In view of properties (a) and (c) above it follows that $|R_{2k}| \leqq I$. Consequently,

$$\lim_{k\to\infty} \frac{R_{2k}}{2k} = 0.$$

It follows from properties (a) and (b) that each bracket in the series

$$K = D_2 + \cdots + (D_{2h} - D_{2h-2}) + \cdots,$$

when written in terms of the quantities c_{ij}, consists of terms from I with various signs attached. Since no term occurs in more than one bracket, it is seen that K converges. Therefore,

$$\lim_{k\to\infty} \frac{T_{2k}}{k} = K = \lim_{k\to\infty} D_{2k},$$

and it follows that $\lim_{k\to\infty} S_{2k} = K/2$. As a consequence of the simple identity

$$S_{2k+1} = \frac{R_{2k+2}}{2k+1} + \frac{T_{2k}}{2k} \frac{2k}{2k+1},$$

it is verified that $\lim_{n\to\infty} S_n = K/2$, which completes the proof of the theorem.

It is evident that the algebraic complement* A_{ij} of each element a_{ij} in the determinant of Theorem I is itself a determinant of the same type. For the complement is obtained by replacing a_{ij} by $+1$, or -1, as the case may be, and by substituting zero for all other elements in the ith row or in the jth column.

If we consider the determinant D' resulting from (3) by the addition of c_{ij} to the element in the ith row and jth column, for every i and j, it follows from Theorem I that D' is summable. Let E represent a determinant obtained from D' by replacing a finite number of its rows (or of its columns) by a bounded set of numbers. Then we could prove that E is summable by the method used in establishing Theorem I.

Let us consider the sequence $(t_n;\ n = 1, 2, \cdots)$ defined by

$$t_{2k-1} = 2k, \qquad t_{2k} = 2k - 1 \quad (k = 1, 2, \cdots), \tag{7}$$

as the normal order for the arrangement of the positive integers. Then a sequence $p = (j_h;\ h = 1, 2, \cdots)$, containing all the positive integers, will be called an *alteration* of the order (7) if, for all values of h sufficiently large, $j_h = t_h$. Thus, for every alteration p, there exists a smallest even index $h = 2m$ such that $j_k = t_k$ if $k \geqq 2m - 1$. Consider the first $(2m - 2)$ terms of p,

$$j_1, \cdots, j_{2m-2}, \tag{8}$$

which give a permutation of the numbers $(1, 2, \cdots, 2m - 2)$. Let us call p an *even* or *odd* alteration according as (8) is obtainable from the order $(1, \cdots, 2m - 2)$ by an even or by an odd number of successive interchanges of neighboring numbers. It is well known* that the evenness or oddness of p is independent of the way in which these interchanges were made.

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* Cf. Kowalewski, loc. cit., pp. 378, 382.

To each alteration p let us make correspond the term

$$P_p = \pm \prod_{k=1}^{2m-2} a_{k,\, j_k} \Big[\prod_{h=m}^{\infty} (-\, a_{2h-1,2h}\, a_{2h,\, 2h-1})\Big],$$

where the $+$ sign is selected if p is even and the minus if p is odd. Suppose that (1) is a determinant of the type E mentioned above. Then it can be proved that

$$2E = \sum_{(all\, p)} P_p, \tag{9}$$

where the summation is extended over all alterations p of the system of positive integers. On the basis of this result, it can then be shown that, for every j and for every k,

$$E = \sum_{i=1}^{\infty} a_{ij} A_{ij} = \sum_{j=1}^{\infty} a_{kj} A_{kj}, \tag{10}$$

where A_{ij} is the algebraic complement of the element a_{ij} in (1). The proofs of (9) and (10) will not be given since the reasoning would be practically identical with that used in the derivation of similar results in the theory of normal infinite determinants.† By use of (9) and (10), it would be possible to develop a theory for the solution of the infinite system of equations

$$\sum_{j=1}^{\infty} a_{ij} x_j = b_i, \qquad (i = 1, 2, \cdots), \tag{11}$$

provided that the determinant (1) formed by the coefficients a_{ij} is of the type D' defined above. A very casual inspection, however, shows that system (11) could easily be transformed into a system of a well known type whose solution could be obtained in terms of normal infinite determinants.

The determinants we have considered were obtained from (3) by simple transformations. If we should take more complicated determinants than (3) as our points of departure we could obtain results similar to those derived above. For example, consider the determinant of order q given by

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[*] Cf. Kowalewski, loc. cit., p. 9.

[†] Cf. Kowalewski, loc. cit., §§ 154, 155.

$$(12)\qquad \left|\begin{array}{lllll}
1 & & & & \\
& \ddots & & & \\
& & 10 & \cdots & \pm 1 \\
& & \;\;10 & & \\
& & & \ddots & \\
& & & & 10
\end{array}\right| \begin{array}{l} \\ \\ (p\text{th row}), \\ \\ \\ (q\text{th row}) \end{array}$$

where all elements not explicitly indicated are zero and where the ambiguous sign in the last column is selected in such a way that the determinant has the value $+1$. In the infinite determinant D of (1) let the elements a_{ij} in the rows of index $mq+1, mq+2, \cdots, mq+q$, for all values $m = 1, 2, \cdots$, be selected as zero, except for those in the columns $mq+1$, $mq+2, \cdots, mq+q$. Choose these q^2 elements as those of the determinant (12). It is easily verified that D is summable to the value p/q. With this determinant D as a starting point instead of the determinant (3), the same methods that were used above suffice to establish a theorem analogous to Theorem I.

The results stated after Theorem I would also have their analogies in the present case. Once more it should be noted that the determinants considered in this paragraph would not enable one to solve any problem in connection with infinite systems of linear equations which could not equally well be solved by the known theory of normal infinite determinants.

3. *Another Type.* An example of the type* of determinants to be considered in the present section is given by (1) in case

$$(13)\qquad a_{ij} = 0, \qquad (i \neq j); \qquad a_{ii} = (-1)^i, \quad (i = 1, 2, \cdots).$$

Then D is summable to the value zero and satisfies the equation

$$(14)\qquad D = \frac{1}{2}(\lim_{n\to\infty} D_{2n+1} + \lim_{n\to\infty} D_{2n}).$$

Another example of this type is given by (1) if

$$(15)\qquad \begin{array}{c} a_{ij} = 0, \qquad (i \neq j); \\ a_{2i-1,\,2i-1} = \alpha, \qquad (a_{2i,\,2i} = \beta,\ i = 1, 2, \cdots;\ \alpha\beta = 1). \end{array}$$

In this case D satisfies (14), where the first limit is α and the second is 1, so that $D = (\alpha + 1)/2$. It is obvious that (15) may be generalized to the case where the main diagonal elements are $\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_1, \cdots, \alpha_k, \cdots$ and where all other elements are zero, under the supposition that

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* The author acknowledges his indebtedness to Professor L. L. Silverman for suggesting the consideration of the determinants treated in this section.

$$\alpha_1 \alpha_2 \cdots \alpha_k = 1. \tag{16}$$

If, under conditions (13), we should add c_{ij} to the element a_{ij} of D, for all (i, j), the resulting determinant would be summable and its value would be given by (14) in case the (c_{ij}) satisfied the condition of Theorem I. The proof of this statement would be almost a repetition of that of Theorem I.

Let D' represent the determinant resulting from (1), under conditions (15), by the addition of c_{ij} to a_{ij}.

THEOREM II. *In the sum I of Theorem* I *let p_k represent the sum of all terms containing exactly k factors c_{ij}. In* (15) *suppose that $|\alpha| > 1$ and that*

$$\sum_{k=1}^{\infty} p_k |\alpha|^k \tag{17}$$

converges. Then, D' is summable and satisfies the relation

$$D' = \frac{1}{2}(\lim_{n\to\infty} D_{2n+1}' + \lim_{n\to\infty} D_{2n}'). \tag{18}$$

The properties of summable sequences make it clear that the theorem will be completely established if we can show that the two limits in (18) exist. Let us consider the convergence of the sequence (D_{2n+1}'). The last two rows of D_{2n+1}' are:

$$\begin{matrix} c_{2n,\,1}\,, & c_{2n,\,2}\,, & \cdots, & c_{2n,\,2n} + \beta, & c_{2n,\,2n+1} \\ c_{2n+1,\,1}, & c_{2n+1,\,2}, & \cdots, & c_{2n+1,\,2n}, & c_{2n+1,\,2n+1} + \alpha. \end{matrix} \tag{19}$$

In expanding D_{2n+1}' by Laplace's Rule,* according to the minors of the last two rows, it is verified that we obtain

$$D_{2n+1}' = D_{2n-1}' + P_{2n+1}, \tag{20}$$

where P_{2n+1} consists of terms containing α, β and factors c_{ij}. Let the expression *frame of a term* of P_{2n+1} refer to the absolute value of the product of those factors c_{ij} entering in the term. The expression P_{2n+1} has the following properties:

(*a*) All terms contain at least one c_{ij} from those in (19) and hence the frames of terms of P_{2n+1} are distinct from those of P_{2h+1} for $h < n$.

(*b*) There are no repetitions among the frames of the terms of P_{2n+1}.

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* Cf. Bôcher, *Introduction to Higher Algebra*, p. 24.

(c) After the relation $\alpha\beta = 1$ has been used, the power of α entering in a term of P_{2n+1} is at most one unit greater than the number of factors c_{ij} in that term. If a power of β remains after use of $\alpha\beta = 1$, the absolute value of the term is increased by neglecting the power of β because $|\beta| < 1$.

It is seen that the convergence of the sequence (D_{2n+1}') is equivalent to that of the series

$$(21) \qquad D_1' + P_3 + P_5 + \cdots + P_{2n+1} + \cdots.$$

Let $\bar{P}_{2n+1}$ represent P_{2n+1} with all of its terms replaced by their absolute values. As a consequence of the properties (a), (b) and (c) it follows that the convergence of (17) implies that of (21) because $\bar{P}_3 + \bar{P}_5 + \cdots \leqq |\alpha| \sum_{k=1}^{\infty} |\alpha|^k p_k$. Hence the sequence (D_{2n+1}') converges. A proof of the same nature as that just given would establish the convergence of the sequence (D_{2n}'). Therefore, Theorem II may be considered completely proved. If we had supposed $|\beta| > 1$, an analogous proof would have shown D' to be summable under the same assumption as was made in Theorem II.

Let E represent the determinant associated with (16) after c_{ij} has been added to the element in row i, column j, for all (i, j). The determinant E is summable if condition (17) is satisfied, where we understand $|\alpha|$ to represent the absolute value of the product of all factors α_h from (16) which satisfy $|\alpha_h| > 1$. The proof of this statement would be similar to that given for Theorem II. E would satisfy the equation

$$(22) \quad E = \frac{1}{k}\left(\lim_{n\to\infty} E_{kn} + \lim_{n\to\infty} E_{kn+1} + \cdots + \lim_{n\to\infty} E_{kn+k-1}\right).$$

Certain interesting questions regarding summable determinants remain unanswered in this note. It may be that Theorem II and its generalization, in connection with determinant E, are true under hypotheses less restrictive than those of this paper. More generally, it would be of interest to know whether a determinant (1), which is summable, remains so if quantities (c_{ij}) are added to its elements, where the c_{ij} satisfy the condition of Theorem I.

THE UNIVERSITY OF MINNESOTA

ALGEBRAIC GUIDES TO TRANSCENDENTAL PROBLEMS.*

BY R. D. CARMICHAEL

1. *Introduction and Historical Sketch.* Up to the present time algebraic guides to transcendental problems have been employed extensively through only a small part of the range of subject matter to which they are so adapted as to yield important characteristic values in suggested theorems and in methods of proving them. This interaction between algebraic and transcendental analysis has attracted greater attention in the theory of integral equations than elsewhere. The relation between the theory of integral and of algebraic equations seems to have been first noticed by Volterra, who pointed out (TORINO ATTI, 1896, pp. 311–323) that a Volterra integral equation of the first kind may be regarded as in a certain sense a limiting form of a system of n linear algebraic equations in n variables as n becomes infinite. It is clear from Volterra's remarks in 1896 that the same is true of the Volterra equation of the second kind, though this fact was not then mentioned explicitly. In 1913 in his *Leçons sur les Équations Intégrales et les Équations Intégro-différentielles,* Volterra brings out in detail (pp. 30–33, 40–52) the connection between the algebraic theory and his equation of the second kind, and less fully (pp. 56 ff.) the connection between the algebraic theory and his equation of the first kind. He indicates (pp. 71 ff.) extensions of the method to systems of integral equations and to equations and systems with multiple integrals, and also (pp. 138 ff.) to the theory of permutable functions. (See also the preface and pp. 33, 102, 117 for remarks on the history of the subject and for references.) We shall set forth the character of the method by a brief indication of the nature of Volterra's treatment of the equation of the second kind.

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* Address as retiring Chairman of the Chicago Section of this Society, Toronto, December 27, 1921. Read for the author by Professor Arnold Dresden.

If in the equation

$$\varphi(x) = u(x) + \int_0^x K(x, \xi)u(\xi)d\xi$$

we replace the integral by the limit to which it is equal by definition, we have the relation

$$\varphi(x) = u(x) + \lim_{n\to\infty} \sum_{i=1}^{n} K(x, \xi_i)u(\xi_i)(\xi_i - \xi_{i-1}),$$

where the points $\xi_0 = 0, \xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n = x$ of the interval $(0, x)$ are so distributed that the greatest difference $\xi_i - \xi_{i-1}$ approaches zero with $1/n$. Approximating to the latter equation is the following:

$$\varphi(x) = u(x) + \sum_{i=1}^{n-1} K(x, \xi_i)u(\xi_i)(\xi_i - \xi_{i-1});$$

and in particular the system

$$\varphi(\xi_s) = u(\xi_s) + \sum_{i=1}^{s-1} a_{si}u(\xi_i), \quad (s = 1, 2, \cdots, n),$$

where $a_{si} = K(\xi_s, \xi_i)(\xi_i - \xi_{i-1})$. Now this algebraic system has a solution in the form

$$u\,(\xi_s) = \varphi(\xi_s) + \sum_{i=1}^{s-1} c_{si}\varphi(\xi_i), \quad (s = 1, 2, \cdots, n);$$

and reciprocally the former system affords the solution of the latter for the $\varphi(\xi_i)$ in terms of the $u(\xi_i)$. By analyzing these two systems Volterra obtains several fundamental relations between them. Then he proceeds heuristically to the limiting forms of these relations after the manner of forming an integral as the limit of a sum, thus obtaining the main principles upon which rests his solution of the given integral equation—these principles then being established *de novo* by methods suggested by the algebraic analysis.

The methods of Volterra may be extended to the case of the Fredholm equation; but an added difficulty arises in the new situation from the fact that the basic algebraic system has now a general determinant depending upon the kernel K rather than the much simpler determinants of the Volterra treatment. It was Fredholm's achievement to see how the method of Volterra could be extended so as to pass from the solution of a system of linear algebraic equations to the solution of the Fredholm integral equation of the second kind. Again in the case of Fredholm, the method was used merely as a heuristic guide for discovering the facts and suitable methods for their proof. With Hilbert arose for integral equations a marked extension of the method through his plan of deducing the results by limiting processes from algebraic propositions as the number of variables becomes infinite.

But the better-known applications to integral equations do not afford the earliest important uses of algebraic guides to transcendental problems. A limiting process having certain points in common with that of Volterra was employed by Cauchy in his proof of the existence of integrals of a system of differential equations, and has been preserved in the lectures of Moigno published in 1844. Considerably simplified and clarified by Lipschitz it stands today under the name of the Cauchy-Lipschitz method (with the extensions of Picard and Painlevé) as one of the principal means for establishing the existence of integrals of differential equations. (See, for instance, Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part 2, pp. 68–74.)

In some respects a closer but still a rather remote analogy with the work of Volterra is afforded by the unpublished method by which Sturm was led to many of the results of his great memoirs of 1836. On page 186 of the first volume of LIOUVILLE'S JOURNAL, Sturm tells us that it was from the solutions of the difference equation

$$L_i u_{i+1} + M_i u_i + N_i u_{i-1} = 0, \quad (i = 0, 1, 2, \cdots),$$

which is nothing more than a somewhat disguised special form of a system of linear algebraic equations, that he was led to the subject of his first memoir of 1836, the memoir in which his more characteristic results for differential equations are to be found; and that the method by which the latter were obtained from the former was one of passing by a limiting process from finite to infinitely small differences. He adds that he had also found for the difference equation properties which are not susceptible of being carried over to differential equations. From these remarks, which he seems never to have elaborated anywhere in his published writings, it is clear that Sturm was led to many of his classic results by a fundamental algebraic guide to the transcendental problems treated.

If one has in mind the suggestive remark of Sturm and the limiting process of Cauchy's existence proof, there is no longer any difficulty in carrying through the limiting process from the difference to the differential equation and of obtaining properties of the solution of the latter directly from those of the former. This was done in an interesting case by Porter in 1902 (ANNALS OF MATHEMATICS (2), vol. 3 (1902), pp. 55–70) more than two years before Hilbert in 1904 took a similar step for integral equations. If we remember that the difference equations used in this analysis are but condensed forms of certain systems of linear algebraic equations, we shall see that the processes which we have usually associated principally with the theory of integral equations were used heuristically in the theory of differential equations long before they were similarly employed for integral equations, and that they were used some years earlier in the former than in the latter in the way of a rigorous passage to the limit.

In Bôcher's Fifth International Congress address and in his *Les Méthodes de Sturm* (see especially chapter 2, pp. 14–42) there is brought out more clearly and more fully than by any of his predecessors the intimate relation that exists between the theory of systems of linear algebraic equations and differential equations, particularly with boundary conditions. Several important theorems for differential systems are obtained as immediate analogues of well known fundamental theorems for algebraic systems, and several intimately related aspects of the two problems are brought out forcibly by the statement of theorems common to the two and identical in their principal characteristics.

It will be observed that all the problems treated in these cases (with the exception of that of Cauchy) are linear in character. This is not accidental; it arises from two facts. Many of the profound phenomena of nature are subject to laws whose expression in mathematical form gives rise to fundamental linear problems of several kinds. Logically the simplest and historically the first to be treated in detail of the linear problems of pure mathematics are those having to do with systems of linear algebraic equations. And these hold the place of greatest importance, both on account of their simplicity and relatively complete development and on account of their suggestiveness in leading the way to transcendental linear problems which emerge from a direct consideration of the natural limiting cases of algebraic systems under the guidance of current problems of transcendental analysis.

My principal purpose in this address is to discuss two types of algebraic theorems and the transcendental problems to whose solution they lead the way. In the one class we shall have oscillation and comparison theorems, and in the other theorems of expansion in orthogonal functions and their generalizations. To one acquainted with the relevant theory of differential and integral equations, it is clear that there is already at hand a large body of doctrine having to do with certain transcendental problems in each of these domains, namely, the classic oscillation and comparison theorems of Sturm for differential equations of the second order, the Sturm-Liouville expansions, expansions by means of the biorthogonal functions arising from Birkhoff's theory of differential systems and their adjoints, and the expansion theorems arising in the theory of integral equations.

The principal algebraic theorems to which I shall direct your attention were conceived in the first place by considering what properties of certain approximating algebraic systems correspond to the properties already established for the transcendental problems which have been treated. It was not difficult to arrive at the corresponding theorems for the special algebraic systems which were involved in certain of these cases; and the theorems once in hand for the special systems were readily extended to fairly general classes of algebraic systems. With this much accomplished, one is in possession of algebraic facts suitable to serve as a guide to a large class of transcendental problems having certain analogies with the problems which suggested the algebraic theorems in the first place. The interaction thus set up between the algebraic and the transcendental problems has a useful power of leading forward to the discovery of important results.

In order to have clearly in mind the nature of the connection between a differential equation and a system of algebraic equations, we shall set up the relation in one of the simplest important cases. A differential equation, with or without boundary conditions, may be realized in an infinite number of ways as the limiting form of an algebraic system, so that there is always room for choice in setting up the system, and in fact need for care that it shall be done in a convenient way. In connection with the equation of second order

$$u''(x) + \varphi(x)u(x) = 0,$$

it is often convenient to employ the approximating equation

$$\frac{u(x+2\delta) - 2u(x+\delta) + u(x)}{\delta^2} + \varphi(x+\delta)u(x+\delta) = 0,$$

which reduces to the differential equation when δ approaches zero, provided that u and φ are subject to appropriate conditions. This equation reduces to

$$u(x) + \{\delta^2\varphi(x+\delta) - 2\}u(x+\delta) + u(x+2\delta) = 0.$$

If we are concerned with the original differential equation when x ranges over an interval (ab), we may take $\delta = (b-a)/n$ where n is an integer. Then, giving to x in the last equation the values a, $a+\delta$, $a+2\delta$, $\cdots$, $a+(n-2)\delta$, we have an algebraic system of $n-1$ equations of the form

$$u(a+i\delta) + a_i u(a+\overline{i+1}\delta) + u(a+\overline{i+2}\delta) = 0, \qquad (i = 0, 1, \cdots, n-2),$$

from which to find the $n+1$ unknown quantities $u(a)$, $u(a+\delta)$, $\cdots$, $u(a+n\delta)$. From properties of the solutions of this algebraic system one can pass back heuristically to properties of the solutions of the differential equation and then can establish these properties by a *de novo* argument suggested by the methods for dealing with the algebraic system. This special case may suggest certain principal characteristics of the general method as applied to differential equations.

The two instances given, the one just developed and that of Volterra, do not by themselves afford an adequate suggestion of the range of applicability of the method. From these cases one can see in part how it works for integral equations and ordinary differential equations. In a similar way it may be brought to bear upon the theory of difference and q-difference equations, both ordinary and partial, and the theory of partial differential equations. Moreover, if we pass from any of these cases by another limiting process of Volterra to such limiting forms as his linear integro-differential equations, or to linear integro-difference and integro-q-difference equations, or to various linear systems combining the properties of these mentioned types of transcendental equations, we shall be able to look upon any one of these directly as a limiting case of an algebraic system under some appropriate method of passing to the limit. In fact, it is probable that certain essential elements of these algebraic guides to transcendental problems can be realized in the case of any transcendental linear problem to which one may be led naturally.

2. *Algebraic Oscillation and Comparison Theorems.* Let us consider a graphical representation of the set of real constants $u_1, u_2, \cdots, u_n$ obtained in the following manner (cf. M. B. Porter, ANNALS OF MATHEMATICS (2), vol. 3 (1901), p. 56). On any convenient horizontal straight line segment, say the points s such that $a \leqq s \leqq b$, let us erect n perpendiculars, two of which are at the ends of the segment, while the other $n - 2$ are evenly or unevenly distributed on the interior of the segment. Let these be marked from left to right by the numbers $1, 2, \cdots, n$; and consider them as analogous to the n coordinate axes of a space of n dimensions. Let the greatest distance between two consecutive axes be called the norm of the system of axes. On the ith axis let us take a point at a distance $|u_i|$ from the original segment, and above it or below it according as u_i is positive or negative. Having done this for each value i of the set $1, 2, \cdots, n$, join by straight line segments the point on each axis but the last to the point on the adjacent axis to the right. We thus obtain a broken line

which we shall call the graphic representation of the point $(u_1, u_2, \cdots, u_n)$ in space of n dimensions, or of the set of constants $u_1, u_2, \cdots, u_n$.

This broken line may be taken also as the graph of a continuous function $u(s)$ of the real variable s on the interval $a \leqq s \leqq b$. We shall say that this function $u(s)$ has been obtained from the set of constants $u_1, u_2, \cdots, u_n$ by linear interpolation with respect to the given n axes. The points at which (values of s for which) this broken line cuts the original line segment (viewed as the axis of s) we shall call indifferently the zeros of the set of constants or of the function $u(s)$.

Let us now suppose that we have a given real-valued single-valued function $v(s)$ of the real variable s, continuous and having a finite number of maxima and minima on $a \leqq s \leqq b$, and a set of n axes formed in the manner already indicated. The graph of this function will cut the n axes in points by means of which we may define as above a linearly interpolated function $\bar{v}(s)$. If the given function $v(s)$ is held the same, and the system of axes is subjected to successive changes, so that the norm of the system decreases and approaches zero, it is clear that the resulting sequence of linearly interpolated functions $\bar{v}(s)$ approaches as a limit the function $v(s)$. The situation thus briefly described is typical of the character of the limiting process by which we shall repeatedly pass from an algebraic system to the corresponding transcendental equation or system.

The most interesting known oscillation and comparison theorems are those which arise in connection with linear homogeneous differential equations of the second order. Certain of the most fundamental properties of such an equation, namely, linearity, homogeneity, and that property in virtue of which the general solution may be expressed linearly in terms of two linearly independent particular solutions, are also fundamental properties of the algebraic system of n equations

$$(1) \qquad \sum_{j=1}^{n+2} a_{ij}x_j = 0, \qquad (i = 1, 2, \cdots, n),$$

in the $n + 2$ unknown quantities $x_1, x_2, \cdots, x_{n+2}$, the matrix of the coefficients of this system being of rank n. Such a system possesses two linearly independent solutions u_i, v_i; and in terms of these the general solution may be written in the form $x_i = cu_i + dv_i$, where c and d are arbitrary constants. From the known theory of the differential equation, and by means of its relation to a particular form of system (1) as exhibited in the first section, we are led to certain properties of the solutions of this general system. We shall now state a few of these properties.

If we denote by Δ_{kl} the determinant of the matrix obtained from the matrix of coefficients in (1) by striking out the kth and lth columns, it may be shown without difficulty that Δ_{kl} and the determinant $u_kv_l - u_lv_k$ are both zero or neither zero, provided that when $\Delta_{kl} = 0$ we do not have the exceptional case in which $\Delta_{ml} = 0 = \Delta_{km}$ for every m of the set $1, 2, \cdots, n + 2$ except k and l. A fairly straightforward argument, based on this elementary result, leads to a proof of the following fundamental theorem.

THEOREM. *Let* $\Delta_{i,\, i+1}$ *for a given range* R *of consecutive values of the integer* i *be of one sign and let* I *denote the interval of the s-axis corresponding to this range of* i *in the sense of the treatment in the first paragraph of this section. Let* u_i *and* v_i *be two linearly independent solutions of the system* (1) *the matrix of whose coefficients is of rank* n*; and let these solutions be extended, by the method of linear interpolation employed above, to the functions* $u(s)$ *and* $v(s)$. *Then on the interval* I *the zeros of* $u(s)$ *and* $v(s)$ *separate each other.*

If one examines the proof (see AMERICAN JOURNAL, vol. 43 (1921), p. 84) by which the foregoing theorem is established, he will see that it depends intimately upon the fact that the determinant $u_{i+1}v_i - u_iv_{i+1}$ is of one sign on R. If one undertakes to formulate a corresponding theorem for the more general system

$$(2) \qquad \sum_{j=1}^{n+h} a_{ij}x_j = 0, \qquad (i = 1, 2, \cdots, n),$$

in the $n + h$ unknown quantities $x_1, x_2, \cdots, x_{n+h}$, where $h \geqq 2$, the matrix of coefficients being again of rank n, he will find the situation in some respects the same as before, but in other (and perhaps more important) respects he will find it far different. If we let D_i denote the determinant of the nth order matrix obtained from the matrix of coefficients in (2) by striking out h consecutive columns beginning with the ith, and if we denote by w_i the determinant of order h,

$$w_i = \begin{vmatrix} x_i^{(1)} & x_{i+1}^{(1)} & \cdots & x_{i+h-1}^{(1)} \\ x_i^{(2)} & x_{i+1}^{(2)} & \cdots & x_{i+h-1}^{(2)} \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ x_i^{(h)} & x_{i+1}^{(h)} & \cdots & x_{i+h-1}^{(h)} \end{vmatrix},$$

formed by means of a fundamental system of solutions $x_i^{(1)}$, $x_i^{(2)}, \cdots, x_i^{(h)}$ of (2), we shall find that for a range R of values of i on which D_i is of one sign it is true that w_i is of one sign; so that in this respect the situation is the same for all values of h.

But if we undertake to proceed further in the direction of a generalization of the theorem stated above, we find, not indeed that our steps are arrested, but that the theorem begins to lose its elegance and simplicity as soon as h is greater than 2, and that the complexity increases rapidly with increase of h. The marked simplicity for the case $h = 2$ is due in large measure to the fact that the expanded determinant w_i has but two terms when $h = 2$. It is possible to put in a variety of forms the complete generalization which emerges, but there seems to be no way in which we can proceed directly to the goal without a surrender of the elegance and simplicity of the theorem. The results which emerge are, however, not entirely without interest; in particular, they appear to point in the direction of theorems for differential equations (for instance) of general order h; but these will necessarily be rather complicated. We shall make no attempt to state any of the results here for general h, either for the case of the algebraic system or that of any of its limiting forms.

If the way of progress by such direct generalization is barred, we shall naturally seek the goal by some other means. Since the difficulty arises primarily from the fact that we have too many linearly independent solutions of (2), that is, too many arbitrary constants in the general solution of (2), let us restrict attention to a particular class of solutions in which the number of arbitrary elements is 2. For the case of the algebraic system (2) the simplest way to do this is to adjoin $h - 2$ additional independent equations of the same type; and this brings us back to the case of system (1). But in the case of a differential or difference equation, for instance, such a means is not directly open to us. However, it is true that the boundary conditions in the transcendental cases are represented in the algebraic system by linear equations (see Bôcher's *Les Méthodes de Sturm*, loc. cit.). The suggestion then is to employ, for a difference or differential equation of order h, boundary conditions (usually $h - 2$ in number) so that there shall be but two arbitrary elements in the general solution subject to these boundary conditions. In § 5, we shall exhibit certain results suggested by these considerations.

In what follows in this section, we shall suppose that the notation in (1) is so chosen that the determinants of the square matrices of orders 1, 2, 3, $\cdots$, n in the lower right-hand corner of the matrix $||a_{ij}||$ are all different from zero. Without loss of generality, they may be taken to be positive, since, if they were not so, this could be brought about by changing the sign of every coefficient in certain of the equations in (1); and therefore we take them to be positive. We *assume* further that the determinants of the square matrices of orders 1, 2, 3, $\cdots$, n in the upper left-hand corner of the matrix $||a_{ij}||$ are all different from zero. Then it is possible to reduce system (1) to a new system

$$x_{i+2} + \alpha_i x_{i+1} + \beta_i x_i = 0, \quad (i = 1, 2, \cdots, n),$$

where α_i and β_i are determinate functions of the original coefficients a_{ij}, and β_i is positive for all values $i = 1, 2, \cdots, n$. On writing $x_i = y_i u_i$, where y_1 and y_2 are positive quantities and

$$y_{2i+1} = \beta_{2i-1}\beta_{2i-3} \cdots \beta_3\beta_1 y_1, \qquad y_{2i} = \beta_{2i-2}\beta_{2i-4} \cdots \beta_4\beta_2 y_2,$$

we obtain for the u_i the relations

$$u_{i+2} + \varphi_i u_{i+1} + u_i = 0, \qquad \varphi_i = \alpha_i \frac{y_{i+1}}{y_{i+2}}, \tag{3}$$

where φ_i is a determinate function of the a_{ij} in (1). A second system of the form (1), under such hypotheses as we have just employed, would reduce to the normal form

$$v_{i+2} + \psi_i v_{i+1} + v_i = 0, \quad (i = 1, 2, \cdots, n). \tag{4}$$

Comparison theorems for the distribution of the zeros of the functions $u(s)$ and $v(s)$, obtained from the constants u_i and v_i by linear interpolation, yield corresponding theorems for the two original systems of form (1). We state a few of the results for the normal forms (3) and (4).

THEOREM. *Let u_i and v_i be solutions of equations* (3) *and* (4), *respectively, and let $u(s)$ and $v(s)$ denote the functions into which they interpolate linearly with respect to a given system of coordinates. If $u(s)$ has consecutive zeros on the μth and $(m + 1)$th intervals, $\mu < m$, then $v(s)$ has a zero between these zeros of $u(s)$ provided that either*

(a) $\varphi_i \leqq \psi_i$, $(i = \mu, \mu + 1, \cdots, m)$, the equality sign not holding for all these values; or

*(b) $\varphi_i = \psi_i$, $(i = \mu, \mu + 1, \cdots, m)$, and the sets of constants u_i and v_i, for $i = \mu, \mu + 1, \cdots, m$, are linearly independent.**

From this several further properties of comparison are readily derived. We state two of them.

Suppose that $u_1 \neq 0$, $v_1 \neq 0$, $\varphi_i \leqq \psi_i$, $(i = 1, 2, \cdots, \nu)$, and that $u_2/u_1 > v_2/v_1$. Then, if $u(s)$ has k zeros on the first ν intervals of the coordinate system, $v(s)$ has at least k zeros on these intervals and the jth of these zeros (in increasing order) of $v(s)$ is to the left of the jth one of $u(s)$.

Let u_1, v_1, u_{k+1}, v_{k+1} be all different from zero and let $u_2/u_1 > v_2/v_1$. Let $u(s)$ and $v(s)$ have the same number (which may be zero) of roots on the first k intervals. Then we have

$$\frac{u_{k+2}}{u_{k+1}} > \frac{v_{k+2}}{v_{k+1}},$$

provided that $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, k$. Other similar theorems may be stated by modifying in certain respects the inequalities in the hypothesis and the conclusion.

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* Compare the related theorem due to M. B. Porter, ANNALS OF MATHEMATICS, (2), vol. 3 (1902), p. 65.

3. *Algebraic Expansion Problems.* The r homogeneous linear algebraic systems in the unknown quantities x,

$$(5)\quad \sum_{j=1}^{n_h} (a_{0hij} + \lambda_1 a_{1hij} + \lambda_2 a_{2hij} + \cdots + \lambda_r a_{rhij}) x_{hj} = 0, \qquad (i = 1, 2, \cdots, n_h),$$

a separate system being formed for each value h of the set $1, 2, \cdots, r$, and the r adjoint systems

$$(6)\quad \sum_{j=1}^{n_h} (a_{0hji} + \lambda_1 a_{1hji} + \cdots + \lambda_r a_{rhji}) y_{hj} = 0, \qquad (i = 1, 2, \cdots, n_h),$$

are consistent (in the sense that each system in each set of r systems has a solution not identically zero) for precisely the same values of the parameters $\lambda_1, \lambda_2, \cdots, \lambda_r$, namely, those values which satisfy the characteristic system of r determinantal equations

$$(7)\quad |a_{0hij} + \lambda_1 a_{1hij} + \cdots + \lambda_r a_{rhij}| = 0, \qquad (h = 1, 2, \cdots, r),$$

where for a given value of h the first member is the determinant of order n_h whose element in ith row and jth column is that which is written out explicitly.

The sets of characteristic values of $\lambda_1, \lambda_2, \cdots, \lambda_r$ for (5) and (6), namely, the sets of solutions of (7), necessarily finite in number if certain exceptional cases are avoided (as we intend they shall be), we shall denote by

$$\lambda_1^{(\rho)}, \lambda_2^{(\rho)}, \cdots, \lambda_r^{(\rho)}$$

for varying values of ρ, the two ordered sets being distinct for two distinct values of ρ. The corresponding solutions of (5) and (6) we shall then denote by

$$x_{hj}^{(\rho)}, y_{hj}^{(\rho)}, \qquad (j = 1, 2, \cdots, n_h;\ h = 1, 2, \cdots, r).$$

If we avoid certain exceptional cases, readily described in terms of the coefficients a in (5) and (6), we then have (for varying ρ and σ) the following relations expressing the fundamental conjugate character of the solutions of (5) and (6), namely:

$$(8)\quad \sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1j_1\cdots i_rj_r} \prod_{h=1}^{r} x_{i_h h}^{(\rho)} y_{hj_h}^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases}$$

where

$$D_{i_1 j_1 \cdots i_r j_r} = \begin{vmatrix} a_{11 j_1 i_1} & a_{21 j_1 i_1} & \cdots & a_{r1 j_1 i_1} \\ a_{12 j_2 i_2} & a_{22 j_2 i_2} & \cdots & a_{r2 j_2 i_2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1r j_r i_r} & a_{2r j_r i_r} & \cdots & a_{rr j_r i_r} \end{vmatrix}.$$

In what follows we shall assume that the mentioned exceptional cases are avoided, so that we shall have situations in which relations (8) are valid.

In order to bring out clearly the nature of these conditions of conjugacy, let us consider the case when $r = 1$. Systems (5) and (6) then take the form

$$(9)\quad \sum_{j=1}^{n} (c_{ij} + \lambda b_{ij}) x_j = 0, \qquad \sum_{j=1}^{n} (c_{ji} + \lambda b_{ji}) y_j = 0, \qquad (i = 1, 2, \cdots, n).$$

The conditions of conjugacy reduce to

$$(10)\qquad \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} x_i^{(\rho)} y_j^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma. \end{cases}$$

In the more special case when $b_{ii} = b_i$, $b_{ij} = 0$ if $j \neq i$, these become

$$(11)\qquad \sum_{i=1}^{n} b_i x_i^{(\rho)} y_i^{(\sigma)} \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma. \end{cases}$$

When $b_i = 1$ for every i these become merely the usual conditions of biorthogonality; and these in turn reduce to the usual conditions of orthogonality in case we have $c_{ij} = c_{ji}$ and the same solutions x and y are taken for the two systems in (9). Thus we see that (8) affords an extensive generalization of a classic elementary relation of wide usefulness.

If m is the number of sets of characteristic values for (5) and (6), and if the set of constants $z_{i_1 i_2 \cdots i_r}$, for i_h varying from 1 to n_h for each h, may be "expanded" in the form

$$(12)\qquad z_{i_1 i_2 \cdots i_r} = \sum_{k=1}^{m} c_k x_{1 i_1}^{(k)} x_{2 i_2}^{(k)} \cdots x_{r i_r}^{(k)},$$

where the c_k are independent of the subscripts i_h, the foregoing properties of conjugacy are available for an immediate determination of the c_k in the form

$$(13)\quad c_k = \frac{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1j_1\cdots i_rj_r} z_{i_1i_2\cdots i_r} \prod_{h=1}^{r} y_{hj_h}{}^{(k)}}{\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r} D_{i_1j_1\cdots i_rj_r} \prod_{h=1}^{r} x_{hi_h}{}^{(k)} y_{hj_h}{}^{(k)}}.$$

It is easy to obtain, in terms of the coefficients a of (5) and (6), broad sufficient conditions for the validity of the "expansion" in (12). The essential simplicity of these formulas is more readily apparent from the special case involved in equations (9) to (11). We assume that for this case m has its usual value n. Then if

$$(14)\qquad z_i = \sum_{k=1}^{n} c_k x_i{}^{(k)}, \qquad (i = 1, 2, \cdots, n),$$

we have for c_k one or the other of the values

$$(15)\qquad \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} z_i y_j{}^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} x_i{}^{(k)} y_j{}^{(k)}}, \qquad \frac{\sum_{i=1}^{n} b_i z_i y_i{}^{(k)}}{\sum_{i=1}^{n} b_i x_i{}^{(k)} y_i{}^{(k)}},$$

according as we have the case of (10) or (11). If $b_i = 1$ the latter reaches the maximum of elegance, at least if equations (9) are then self-adjoint and the same solutions x and y of the two systems are taken.

In the foregoing expansion formulas the number of subscripts on the "function" to be expanded in terms of the "functions," x is equal to the number of parameters λ involved in the original system of algebraic equations. When we proceed to the transcendental limiting cases we shall see that these algebraic results are in the form best suited to applications to equations involving functions of one variable. In order to obtain a form suitable as the heuristic guide in problems involving functions of more than one variable in the original equations, it is desirable to look upon our expansion formulas in a way slightly different from that in evidence in the foregoing work. We can best bring out what is needful by considering expansion (14). Let us suppose that n is the product $\mu\nu$ of two integers. Let us replace the subscript i, running over the set $1, 2, \cdots, n$, by the double subscript ij where i

runs over the set 1, 2, $\cdots$, μ and j over the set 1, 2, $\cdots$, ν. Then equations (14) become

$$(16)\quad z_{ij} = \sum_{k=1}^{\mu\nu} c_k x_{ij}^{(k)}, \qquad (i = 1, 2, \cdots, \mu;\ j = 1, 2, \cdots, \nu).$$

The coefficients c_k are of course determined by the same methods as before but the formulas are modified through the replacing of single summations by double summations. It is clear that we may pass in a similar way to multipartite instead of bipartite subscripts. This obvious remark concerning a change from one subscript to more than one subscript entails important consequences in a great variety of expansion problems.

With a given set of n axes on an interval $a \leqq s \leqq b$, as in the opening paragraphs of § 2, let us extend by linear interpolation the n sets $x_i^{(k)}$, $k = 1, 2, \cdots, n$, and the set z_i of equation (14); and let $x^{(k)}(s)$, $k = 1, 2, \cdots, n$, and $z(s)$ denote the functions of the continuous variable s so obtained. Then it is easy to prove that

$$(17)\qquad z(s) = \sum_{k=1}^{n} c_k x^{(k)}(s),$$

where the coefficients c_k have the same values as in (14). Thus we pass from "expansions" of sets of constants to interesting expansions of a particular class of functions of a continuous variable.

Let us consider the like matter for functions of two subscripts and expansions of the form (16). For the representation of u_{ij} for $i = 1, 2, \cdots, \mu$ and $j = 1, 2, \cdots, \nu$, we shall start from ν parallel planes, one for each value of j. In each of these we use for s the same range $a \leqq s \leqq b$ and place the planes in order 1, 2, $\cdots$, ν for j, each directly in front of the preceding one and arrange the vertical axes, so that the vertical axes in any one plane are (point for point) the orthogonal projections on that plane of the axes in any other plane. Next, for fixed j we extend the u_{ij} to $u_j(s)$ by linear interpolation as before. Then for each value of s we connect the points $u_j(s)$, $j = 1, 2, \cdots, \nu$, by straight-line segments joining the consecutive points. We thus get a sort of broken surface affording the graphical representation of a function $u(s, t)$, gotten (we may say) by linear interpolation from the points u_{ij}. We call this the graphical representation of the set of constants u_{ij} with respect to the given system of axes. Let $c \leqq t \leqq d$ denote the range of t in this representation, this range evidently depending on the positions of the ν planes employed in setting up the graphical representation.

Let us replace the subscripts i, j in the functions in (16) by the continuous variables s, t in accordance with the method just indicated. Then it is easy to prove that we have

$$z(s, t) = \sum_{k=1}^{\mu\nu} c_k x^{(k)}(s, t), \qquad (a \leqq s \leqq b,\ c \leqq t \leqq d), \tag{18}$$

where the coefficients c_k have the same values as in (16). Similar (but more complicated) extensions may be associated with (12) so that we come through to a relation of the form

$$z(s_1, s_2, \cdots, s_r) = \sum_{k=1}^{m} c_k x_1^{(k)}(s_1) x_2^{(k)}(s_2) \cdots x_r^{(k)}(s_r), \tag{19}$$

where the coefficients c_k have the same values as in (12) and the functions involved depend on continuous variables. Further generalizations may also be made in (12) and in the last formula by replacing one or more of the subscripts i_h in (12) by multipartite subscripts and by linear interpolation in the resulting relations.

In the case of the principal transcendental expansion problems the foregoing expansions in a finite number of terms are replaced by expansions in an infinite number of terms so that difficult questions of convergence arise. In one of the most important of these transcendental problems, namely, that treated by Birkhoff (TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), pp. 373–395), the convergence questions are dealt with by the aid of a contour integral by means of which the sum of any finite number of terms of the series is readily expressed. Similar contour integrals exist for representing the sum of a finite number of terms of the series in (12) or (14). Let us consider the general case. For any given value of h let $\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the determinant in the first member of (7) and let $\Delta_{hij}(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the cofactor of the element in the

ith row and jth column of this determinant. We assume for the present purpose that systems (5) and (6) are so restricted that the solutions of each for characteristic values of the λ's is unique (except for a constant factor) and that for each h

$$\lim_{\lambda_1=\lambda_1^{(k)},\cdots,\lambda_r=\lambda_r^{(k)}} \frac{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\lambda_h - \lambda_h^{(k)}}$$

exists and is finite, where $\lambda_1^{(k)}, \cdots, \lambda_r^{(k)}$ is any set of characteristic values for (5) and (6). Then, if we avoid a certain exceptional condition not arising in the (most important) case $r = 1$, we have for the kth term of the expansion of $z_{t_1 t_2 \cdots t_r}$ afforded by (12) the value

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^r \int_{\Gamma_{1k}} \int_{\Gamma_{2k}} \cdots \int_{\Gamma_{rk}} \left\{ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \right.$$

$$\left. \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1 j_1 \cdots i_r j_r} z_{i_1 i_2 \cdots i_r} G_{j_1 t_1 \cdots j_r t_r} \right\} d\lambda_1 d\lambda_2 \cdots d\lambda_r,$$

where Γ_{hk} $(h = 1, 2, \cdots, r)$ is a contour in the λ_h-plane about the point $\lambda_h^{(k)}$, containing in its interior no other characteristic value of λ_h, and where

$$G_{j_1 t_1 \cdots j_r t_r} = \prod_{h=1}^{r} \frac{\Delta_{h j_h t_h}(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)}.$$

If we replace the contours Γ_{hk} $(h = 1, 2, \cdots, r)$ by Γ_h, a contour which includes within it all the characteristic values of λ_h, and perform the same multiple integration about such contours, we shall have the value of the function $z_{t_1 t_2 \cdots t_r}$. It is clear that we may form similarly the contour integral for any given partial sum of the series for $z_{t_1 t_2 \cdots t_r}$ in (12). For the case of systems (9) we have the equations

$$z_t = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\Delta_{jt}(\lambda)}{\Delta(\lambda)} b_{ji} z_i d\lambda, \qquad (t = 1, 2, \cdots, n),$$

where Γ is a contour in the λ-plane inclosing all the characteristic values for (9), and where $\Delta(\lambda)$ is the determinant of the coefficients in the first member of (9) and $\Delta_{jt}(\lambda)$ is the cofactor of the element in the jth row and tth column of $\Delta(\lambda)$.

For use in connection with expansion problems involving difference equations, it is desirable to observe that several of the important formal properties of (5) and (6) are preserved in the case of infinite systems in which appropriate conditions of convergence are realized. Thus in (5) and (6) we may allow some or all of the numbers $n_1, \cdots, n_r$ to become infinite; for the sake of simplicity we suppose that all of them become infinite. The system (7) is replaced by a transcendental system having, in the cases which interest us, an infinite number of solutions. Subject to suitable limitations for procuring convergence, we may proceed as before and thus derive a new form of (8) in which the change is that of replacing each of the numbers $n_1, \cdots, n_r$ by ∞, and a new form of (12) in which m is replaced by ∞ and the relations are to be valid for each i_h ranging over the set $1, 2, 3, \cdots$, the expressions for the coefficients c_k $(k = 1, 2, 3, \cdots)$ having the form which results from that in (13) on replacing $n_1, \cdots, n_r$ each by ∞. Analogues of many of the additional foregoing results persist. In the formulas thus obtained we have a useful heuristic guide to certain expansion problems in the theory of difference equations, a part of which we shall presently indicate.

4. *Transcendental Expansion Problems.* Let us consider a fixed interval $a \leqq s \leqq b$ of the real s-axis, and let us associate with certain points of this interval the discrete values of i and j in equations (5) and (6). For the hth system in either (5) or (6), we take on (ab) a set of points n_h in number including the points a and b; and we interpolate x_{hj} into a function $x_h(s)$ by the method of linear interpolation described above. Similarly a_{khij} is interpolated into a function $a_{kh}(s, t)$. Equations (5) and (6) may now be looked upon as establishing relations among the functional values of these functions of s and t at certain points only of the axes of s and t; and so of defining the solution functions at these points and these alone, their definitions being completed by the methods of interpolation agreed upon.

To these systems we now apply certain limiting processes, allowing the numbers n_h (or at least a part of them) to increase indefinitely. As they increase, the functions $x_h(s)$ and $a_{kh}(s, t)$ pass through a corresponding sequence of changes. If the processes involved lead to replacing the original equations by well-defined limiting equations and their solutions by well-defined functions, we have in the process a suggestion of a heuristic guide to probable solutions of the limiting problems and to certain probable fundamental properties of these, together with intimations as to how they shall be established.

Usually we shall require that the distribution of basic points on the interval (ab) of the s-axis shall undergo change in such way that the norm of the distribution shall approach zero. By such processes one may realize, for instance, integral equations and differential equations as limiting cases. Among the characteristic results, for the finite case, which persist after certain limiting operations of this type have been performed are those relating to conjugacy, expansions, and representation of the latter by contour integrals. Without going into details, we may state the general limiting forms of certain equations, special cases of which we shall have occasion to consider. Equations (8), (12), (13) pass into the following limiting forms:

$$\int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \prod_{h=1}^{r} x_h^{(\rho)}(s_h) y_h^{(\sigma)}(t_h) ds_1 dt_1 \cdots ds_r dt_r \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{20}$$

$$z(s_1, s_2, \cdots, s_r) = \sum_{k=1}^{\infty} c_k x_1^{(k)}(s_1) x_2^{(k)}(s_2) \cdots x_r^{(k)}(s_r), \tag{21}$$

$$c_k = \frac{\int_a^b \int_a^b \cdots \int_a^b z(s_1, s_2, \cdots, s_r) D(s_1, t_1, \cdots, s_r, t_r) y_1^{(k)}(t_1)\, y_2^{(k)}(t_2) \cdots y_r^{(k)}(t_r) ds_1 dt_1 \cdots ds_r dt_r}{\int_a^b \int_a^b \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \prod_{h=1}^{r} x_h^{(k)}(s_h) y_h^{(k)}(t_h) ds_1 dt_1 \cdots ds_r dt_r}. \tag{22}$$

Corresponding to (10), (11), (14), (15) we have special cases of importance which indicate more clearly the essential simplicity of the formulas, namely,

$$\int_a^b \int_a^b b(t, s) x^{(\rho)}(s) y^{(\sigma)}(t) ds dt \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{23}$$

$$\int_a^b b(s) x^{(\rho)}(s) y^{(\sigma)}(s) ds \begin{cases} = 0 \text{ if } \rho \neq \sigma, \\ \neq 0 \text{ if } \rho = \sigma, \end{cases} \tag{24}$$

$$z(s) = \sum_{k=1}^{\infty} c_k x^{(k)}(s), \qquad (a \leqq s \leqq b), \tag{25}$$

$$c_k = \frac{\int_a^b \int_a^b b(t, s) z(s) y^{(k)}(t) ds dt}{\int_a^b \int_a^b b(t, s) x^{(k)}(s) y^{(k)}(t) ds dt} \tag{26}$$

or

$$\frac{\int_a^b b(s) z(s) y^{(k)}(s) ds}{\int_a^b b(s) x^{(k)}(s) y^{(k)}(s) ds} .$$

If for the last form of c_k we have $c(s, t) = c(t, s)$, so that $y^{(k)}(s)$ may be taken equal to $x^{(k)}(s)$, and if $b(s) \equiv 1$, we have in (25) and the latter form of c_k in (26) formulas for the formal expansion of an "arbitrary" function of a single variable in terms of orthogonal functions of that variable; so that all the formulas (20) to (26) are generalizations of classic relations in the expansion of functions.

In these formulas the number of variables in the function z to be expanded is equal to the number of parameters involved in the problem. This correspondence is not essential. The desired extension can best be brought out by starting from the particular expansion (16). By linear interpolation we first obtain (18). Then we may proceed to the limiting case in such wise that μ and ν simultaneously approach infinity, the norm of the corresponding distributions of points on the s-axis and the t-axis approaching zero. We are thus led heuristically to an expansion of the form

$$z(s, t) = \sum_{k=1}^{\infty} c_k x^{(k)}(s, t), \tag{27}$$

where the x's are now solutions of the limiting problem. It is easy to see that the properties of conjugacy are maintained formally and that we may therefore readily determine the coefficients c_k. We may also proceed from (18) to the limit in another way, namely, by holding μ fixed and allowing ν to become infinite as before; we are thus led to expansions of the form

$$z_i(s) = \sum_{k=1}^{\infty} c_k x_i^{(k)}(s), \quad (i = 1, 2, \cdots, \mu), \tag{28}$$

for expanding a system of μ given functions in terms of μ sets of functions, the coefficients c_k of the expansion being the same for each of the given functions. This remarkable type of expansion I saw first in the manuscript of Dr. C. C. Camp's dissertation which he was kind enough to allow me to read; it occurred there in connection with a particular system of two linear differential equations of the first order.* In (27) we have an expansion of the type which arises in the theory of partial differential equations in two independent variables. In (28) we have the type of expansions which arises in the theory of n linear differential equations of the first order. It is clear that the type of extension employed in this paragraph for the case of one parameter may be utilized in a variety of ways in connection with problems involving r parameters.

Let us consider the results to which this heuristic guide leads us in the case of the adjoint differential systems

$$(29)\qquad \frac{dy_i}{dx} = \sum_{j=1}^{n} (a_{ij} + \lambda\alpha_{ij})y_j, \qquad (i = 1, 2, \cdots, n),$$

$$(30)\qquad \frac{dz_i}{dx} = \sum_{j=1}^{n} (- a_{ji} - \lambda\alpha_{ji})z_j, \qquad (i = 1, 2, \cdots, n).$$

If for fixed i we multiply these respectively by z_i and y_i, add the resulting equations member by member, sum as to i from 1 to n, and then integrate from a to b (a range in which the coefficients are assumed to be continuous) we have

$$(31)\qquad [y_1z_1 + y_2z_2 + \cdots + y_nz_n]_{x=a}^{x=b} = 0.$$

The first member of this relation is a non-singular bilinear form in the two sets of $2n$ variables each,

$$(32)\qquad y_1(a),\ y_2(a),\ \cdots,\ y_n(a),\ y_1(b),\ \cdots,\ y_n(b); \quad z_1(a),\ \cdots,\ z_n(a),\ z_1(b),\ \cdots,\ z_n(b).$$

It can be written in an infinite number of ways in the form

$$(33)\qquad [y_1z_1 + \cdots + y_nz_n]_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} Y_i(y)Z_i(z),$$

where $Y_i(y)[Z_i(z)]$ is a set of $2n$ linearly independent homogeneous linear functions of the $2n$ variables $y[z]$ of (32). With (29) and (30), respectively, we associate the boundary conditions

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* Since this was written a paper by A. Schur has appeared dealing with this problem; see MATHEMATISCHE ANNALEN, vol. 82 (1921), pp. 213–236. It contains (p. 214) a reference to a special case of the problem treated by Hilbert (GÖTTINGER NACHRICHTEN, 1906, pp. 474–480). This I had not seen before.

$$Y_i(y) = 0, \qquad (i = 1, 2, \cdots, n), \tag{34}$$

$$Z_i(z) = 0, \qquad (i = n + 1, \cdots, 2n). \tag{35}$$

Then the characteristic values λ for problem (29), (34) are the same as those for problem (30), (35). We suppose that the conditions are set up in such a way that the number of these characteristic values is infinite. We denote the characteristic values and the corresponding solutions by $\lambda^{(k)}$, $y_i^{(k)}$, $z_i^{(k)}$, $(k = 1, 2, 3, \cdots)$. Corresponding to (20), (21), (22) we now have the relations

$$\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ji} y_i^{(k)} z_j^{(l)} \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f_i(x) = \sum_{k=1}^\infty c_k y_i^{(k)}, \qquad (i = 1, 2, \cdots, n),$$

$$c_k = \frac{\displaystyle\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ji} f_i z_j^{(k)} dx}{\displaystyle\int_a^b \sum_{i=1}^n \sum_{j=1}^n \alpha_{ji} y_i^{(k)} z_j^{(k)} dx}, \qquad (k = 1, 2, 3, \cdots).$$

If we apply to systems (29) and (30) a limiting process often employed by Volterra we are led to adjoint integro-differential equations of the form

$$\frac{\partial y(x, s)}{\partial x} = \int_\alpha^\beta \{u(x, s, t) + \lambda v(x, s, t)\} y(x, t) dt,$$

$$\frac{\partial z(x, s)}{\partial x} = \int_\alpha^\beta \{- u(x, t, s) - \lambda v(x, t, s)\} z(x, t) dt,$$

where the range of variation of x is from a to b while that of s and t is from α to β. Corresponding to the last three equations of the preceding paragraph we now have the following:

$$\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s)\, y_k(x, s)\, z_l(x, t)\, ds\, dt\, dx \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f(x, s) = \sum_{k=1}^\infty c_k y_k(x, s),$$

$$c_k = \frac{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s) f(x, s)\, z_k(x, t)\, ds\, dt\, dx}{\int_a^b \int_\alpha^\beta \int_\alpha^\beta v(x, t, s)\, y_k(x, s)\, z_k(x, t)\, ds\, dt\, dx}, \qquad (k = 1, 2, \cdots),$$

where $y_k(x, s)$, $z_k(x, s)$ denote the solutions corresponding to the characteristic value λ_k of the set $\lambda_1, \lambda_2, \cdots$ of distinct characteristic values.

The same procedure may be applied with equal facility to the adjoint systems of integro-differential equations

$$\frac{\partial y_i(x, s)}{\partial x} = \sum_{j=1}^{n} (a_{ij} + \lambda\alpha_{ij})\, y_j(x, s) + \int_a^\beta \sum_{j=1}^{n} [\rho_{ij}(x, s, t) + \lambda\sigma_{ij}(x, s, t)]\, y_j(x, t)\, dt,$$

$$\frac{\partial z_i(x, s)}{\partial x} = \sum_{j=1}^{n} (- a_{ji} - \lambda\alpha_{ji})\, z_j(x, s) + \int_a^\beta \sum_{j=1}^{n} [- \rho_{ji}(x, t, s) - \lambda\sigma_{ji}(x, t, s)]\, z_j(x, t)\, dt,$$

for $i = 1, 2, \cdots, n$, and to various generalizations and extensions of them. In this way emerge formal properties of various types of expansions arising in connection with differential and integro-differential equations. The problems may likewise be set up with equal facility for the case of an equation or system of equations of any order with respect to differentiation instead of merely for the first order as in the foregoing problems. Furthermore, one can treat equally well a system with r parameters λ involved in a way analogous to that observed in connection with the algebraic problem defined at the beginning of § 3. The formulas necessarily become more complicated, but the fundamental guiding ideas are unmodified; the algebraic theory indicates the whole procedure and suggests the principal results as limiting cases of the algebraic propositions.

The way in which the corresponding problem may be set up for the difference equation will be indicated by a very brief statement. Let us consider the adjoint systems of difference equations

$$(36) \quad u_i(x + 1) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda\psi_{ij})\, u_j(x), \qquad (i = 1, 2, \cdots, n),$$

$$(37) \quad v_i(x) - v_i(x + 1) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda\psi_{ji})\, v_j(x + 1), \qquad (i = 1, 2, \cdots, n),$$

where φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and vanish there to an order at least as high as the second. We confine attention to those solutions alone which have the property that each function in a solution approaches a (finite) limiting value as x approaches infinity along any ray from the origin exclusive of the negative axis of imaginaries or along any line proceeding to the right parallel to the axis of reals.

If we multiply (36) through by $v_i(x+1)$ and (37) through by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n} \Delta\{u_i(x)v_i(x)\} = 0. \tag{38}$$

If the real part of a is sufficiently large we have u_i and v_i analytic at every point x whose real part is not less than the real part of a. Hence in (38) we may sum as to x from a to infinity, where x runs over the values a, $a+1$, $a+2$, $\cdots$; thus we have

$$\sum_{i=1}^{n} \{u_i(\infty)\, v_i(\infty) - u_i(a)\, v_i(a)\} = 0. \tag{39}$$

Let us now suppose that adjoint homogeneous linear boundary conditions, implying (39), are set up on the $u_i(\infty)$, $u_i(a)$ and on the $v_i(\infty)$, $v_i(a)$ similar to conditions (34) and (35) in a similar problem above and let us suppose that we have the infinite set of characteristic values and corresponding solutions $\lambda^{(k)}$, $u_i^{(k)}$, $v_i^{(k)}$, $(k = 1, 2, 3, \cdots)$. The fundamental formulas for the expansion problem thus arising are the following:

$$\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t)\, u_i^{(k)}(a+t)\, v_j^{(l)}(a+1+t) \begin{cases} = 0 \text{ if } k \neq l, \\ \neq 0 \text{ if } k = l, \end{cases}$$

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad (i = 1, 2, \cdots, n),$$

$$c_k = \frac{\displaystyle\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t) f_i(a+t)\, v_j^{(k)}(a+1+t)}{\displaystyle\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(a+t)\, u_i^{(k)}(a+t)\, v_j^{(k)}(a+1+t)}, \qquad (k = 1, 2, 3, \cdots).$$

If a classic limiting process of Volterra is applied to systems (36) and (37), they go over into adjoint integro-difference equations analogous to the integro-differential equations treated above; and these may be generalized to related systems just as we generalized the corresponding problem for integro-differential equations. In all these cases the properties of conjugacy and the formal results for expansions persist in the form naturally to be expected. Moreover, corresponding and closely similar theories exist for q-difference and integro-q-difference equations.

In view of the basic algebraic theory and the transcendental problems already treated it is clear that there must exist in the theory of integral equations expansion problems involving r parameters not only in the classic case when $r = 1$ but also in the general case when r is any positive integer. Moreover, if we think of the several types of expansion problems—those for differential, difference, q-difference, integral, integro-differential, integro-difference, and integro-q-difference equations—in intimate connection with the basic algebraic theory, it becomes apparent that the case of r parameters (for $r > 1$) is not confined to a set of r equations of the same sort. There is nothing to prevent one subset of the basic algebraic equations from proceeding to differential equations as limiting forms, another to difference equations, another to integral equations, another to q-difference equations, and so on. Thus we can see beforehand that we may formulate the expansion problem for a variety of mixed systems. One is in fact led naturally to such systems in the consideration of certain integro-differential, integro-difference, and integro-q-difference equations. We shall not take space to treat any of these mixed systems, preferring rather to exhibit briefly the nature of the problem for partial differential equations. From these one may proceed naturally to related integro-differential equations. Similar problems may be formulated for partial difference and integro-difference equations.

Let us consider the adjoint partial differential equations

$$L(u) + \lambda L_1(u) = 0, \qquad M(v) + \lambda M_1(v) = 0,$$

where

$$L(u) \equiv a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u,$$

$$L_1(u) \equiv p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + ru,$$

the symbols $u, v, a, b, c, \alpha, \beta, \gamma, p, q, r$ denoting functions of the independent variables x and y continuous and suitably differentiable in the square $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$. It is easy to show the existence of an identity of the form

$$\int_0^1\int_0^1[v\{L(u)+\lambda L_1(u)\}-u\{M(v)+\lambda M_1(v)\}]dxdy \equiv \int_0^1 Bdt,$$

where B is a sort of bilinear form in the functions u and v and their first derivatives, the arguments of the functions being suitably restricted. If suitable boundary conditions are set up for the u-problem and for the v-problem so that $B \equiv 0$ in virtue of the boundary conditions and so that we have the infinite set of characteristic values and corresponding solutions λ_k, u_k, v_k, $(k = 1, 2, 3, \cdots)$, then we have the following fundamental formulas:

$$\int_0^1\int_0^1 u_i M_1(v_j)\, dx\, dy \begin{cases} = 0 \text{ if } i \neq j, \\ \neq 0 \text{ if } i = j, \end{cases}$$

$$f(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y),$$

$$c_k = \frac{\int_0^1\int_0^1 f(x, y)\, M_1(v_k)\, dx\, dy}{\int_0^1\int_0^1 u_k(x, y)\, M_1(v_k)\, dx\, dy}, \quad (k = 1, 2, 3, \cdots).$$

These results are readily carried over to partial differential equations of other forms and to the case of much more general regions than the square over which we have integrated in this particular instance.

5. *Transcendental Oscillation and Comparison Theorems.* The fundamental algebraic oscillation theorem in the earlier part of § 2 has several limiting forms of interest. We consider first those for homogeneous linear differential equations. The result is classic for equations of the second order: the zeros of two linearly independent solutions of such an equation separate each other throughout any interval containing no singular point of the equation. We shall now state one

extension of this result to equations of order n, $n > 2$. Such an equation we write in the general form

$$u^{(n)} + p_1 u^{(n-1)} + \cdots + p_{n-1} u' + p_n u = 0, \tag{40}$$

where the superscripts refer to differentiation and where the coefficients p are real-valued, single-valued, continuous functions of the real variable x on the interval $(a\ b)$. Since this involves an n-fold infinitude of solutions we shall require boundary conditions to restrict the permissible solutions to a two-fold infinitude linearly dependent on two linearly independent solutions (this being done so that the new theorem shall indeed be a direct limiting form of the algebraic theorem referred to). Suitable boundary conditions may be expressed by means of Stieltjes integrals in the form

$$\sum_{j=1}^{\nu} \int_a^b L_{ij}(u)\, d\psi_{ij}(x) = 0, \tag{41}$$

$$(i = 1, 2, \cdots, n-2;\ \nu = \text{positive integer}),$$

where the $\psi_{ij}(x)$ are functions of bounded variation on (ab) and the $L_{ij}(u)$ denote homogeneous linear expressions in u, u', $\cdots$, $u^{(n-1)}$. As a special case we have conditions which reduce to the following: $u(a) = 0, u'(a) = 0, \cdots, u^{(n-3)}(a) = 0$.

By aid of a fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ of solutions of (40) we define constants λ through the formulas

$$\sum_{k=1}^{\nu} \int_a^b L_{ik}(\bar{u}_j)\, d\psi_{ik}(x) = \lambda_{ij}, \qquad \begin{pmatrix} i = 1, 2, \cdots, n-2 \\ j = 1, 2, \cdots, n \end{pmatrix},$$

and then the determinant $D(x)$,

$$D(x) \equiv \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2,\,1} & \lambda_{n-2,\,2} & \cdots & \lambda_{n-2,\,n} \end{vmatrix}.$$

The zeros of $D(x)$ are independent of the choice of the fundamental system by means of which they are defined; that is, they depend only on (40) and (41). These zeros we shall call the *special points* of (ab) for the problem (40), (41). Then we have the following theorem:

On any interval of (ab) containing no special points for the *problem* (40), (41) *the zeros of any two linearly independent solutions of* (40), (41) *separate each other.*

The foregoing result may be applied to the case of any two linearly independent solutions of (40) without reference to any *preassigned* boundary conditions. For this purpose we associate with any two linearly independent solutions u_1, u_2 of (40) a set of boundary conditions capable of representation in the form (41) and having the property that the solutions of (40) and the determined conditions (41) are those functions and those alone which are linearly dependent upon u_1 and u_2. As a simple example of such associated boundary conditions we have those determined as follows: Let the initial constants for u_1 and u_2 at a point $x = \alpha$ of (ab) be

$$u_i^{(k)}(\alpha) = \rho_{ik}, \qquad (k = 0, 1, 2, \cdots, n-1;\ i = 1, 2).$$

Let the coefficients σ_{ij} be so chosen that the equations

$$\sigma_{i0}u(\alpha) + \sigma_{i1}u'(\alpha) + \sigma_{i2}u''(\alpha) + \cdots + \sigma_{i,\,n-1}u^{(n-1)}(\alpha) = 0, \qquad (i = 1, 2, \cdots, n-2),$$

have those solutions and those alone which may be written in the form $u^{(k)}(\alpha) = a_1\rho_{1k} + a_2\rho_{2k}$, $(k = 0, 1, \cdots, n-1)$, where a_1 and a_2 are arbitrary constants. Having thus introduced suitable boundary conditions restricted by means of the given solutions u_1 and u_2, we may define the determinant $D(x)$, and hence the special points, in the way indicated for the preceding case. By means of these points we may divide the interval (ab) into segments on the interior of each of which the zeros of u_1 and u_2 separate each other in accordance with the foregoing theorem.

As a second case, let us consider the difference equation

$$L(x)\,u(x) + M(x)\,u(x+1) + N(x)\,u(x+2) = 0$$

in which all the indicated functions are real-valued, single-valued, continuous functions of the real variable x for $x \geqq \alpha$, and $L(x)$ and $N(x)$ are both of one and the same sign for $x \geqq \alpha$. Let $u_1(x)$ and $u_2(x)$ be a fundamental system of solutions of this equation and let a $(a \geqq \alpha)$ be a point for which $w(a) \neq 0$ where $w(x) = u_1(x)\,u_2(x+1) - u_1(x+1)\,u_2(x)$. Let $\bar{u}_i(x)$ be the function obtained by linear interpolation from the set of constants $u_i(a)$, $u_i(a+1)$, $u_i(a+2)$, $\cdots$, with respect to a system of coordinate axes obtained by drawing lines perpendicular to the x-axis through the points $a, a + 1, a + 2, \cdots$. Let the zeros of $\bar{u}_i(x)$ on the range $a \leqq x < \infty$ be called the characteristic points of $u_i(x)$ with respect to a. Then we have the following theorem analogous to that of Sturm for a second order differential equation: *The characteristic points of* $u_1(x)$ *and* $u_2(x)$ *with respect to* a *separate each other.*

This result admits of extension to a system formed of a difference equation of order n, $n > 2$, and $n - 2$ boundary conditions of a certain general sort restricting the simultaneous solutions to a two-fold infinitude. Similar results may also be obtained for q-difference equations. In fact, those for difference and q-difference equations are both special cases of like results for a rather general class of functional equations including difference and q-difference equations as special cases; but we shall not here take the space necessary to set forth these more general results. The comparison theorems which follow suggest their nature. The second theorem of § 2 has as a limiting case a theorem which is essentially equivalent to the following classic Sturmian theorem of comparison.

Let us consider the two differential equations

$$\frac{d}{dx}(K_1u') - G_1u \quad = 0, \tag{42}$$

$$\frac{d}{dx}(K_2u') - G_2(u) = 0, \tag{43}$$

in which K_1, K_2, G_1, G_2 are functions which are continuous throughout the interval (ab) defined by the inequalities $a \leqq x \leqq b$ and in this interval satisfy the relations

$$0 < K_2 \leqq K_1, \qquad G_2 \leqq G_1. \tag{44}$$

Moreover, let u_1 and u_2 be solutions of (42) and (43), respectively, neither of which is identically zero in (ab). Then if x_1 and x_2 are any two consecutive zeros of u_1 in (ab) there is at least one zero of u_2 in the interior of the interval (x_1x_2) provided either that at least one equality sign in (44) fails to hold at every point of the interval (x_1x_2), or that u_1 and u_2 are linearly independent in (x_1x_2).

The two results stated in the paragraphs following the theorem referred to in § 2 have as limiting cases two theorems which are also classic in the theory of Sturm. The algebraic theorems were indeed suggested by these Sturmian theorems. The latter, with considerable loss of elegance, have been extended to homogeneous linear differential equations of general order k (ANNALS OF MATHEMATICS (2), vol. 19 (1918), pp. 159–171). It is possible to extend the general algebraic results of the latter part of § 2 to the analogous algebraic case, namely, the case of algebraic systems with k linearly independent solutions; but the results lack (in some respects) the desired elegance. From them one may in turn obtain corresponding properties of a certain class of functional equations. We content ourselves with giving some of these results for the most interesting case, namely, that in which the equations have just two independent solutions.

Let us consider the substitution $x' = S_x$, denoting its nth power by $x' = S_x^n$. Let it be such that there exists an open interval I of the real x-axis, such that $\lim S_{x_0}^n = \beta$ for every x_0 of I, β being an end-point of I and the limit being approached monotonically. Then consider the functional equations

$$u(S_x^2) + \varphi(x)u(S_x) + u(x) = 0, \tag{45}$$

$$v(S_x^2) + \psi(x)v(S_x) + v(x) = 0, \tag{46}$$

in which $\varphi(x)$ and $\psi(x)$ are real-valued, single-valued, continuous functions of the real variable x on the interior of the interval I. Suppose, furthermore, that S_x is such that each of these equations has a fundamental system of solutions consisting of two functions which are real-valued, single-valued, and continuous on the open interval I. (In case $S_x \equiv x + 1$ and I is the interval $\alpha < x < \infty$, our equations are ordinary difference equations; in case $S_x = qx$, q being real and greater than unity, and I is the interval $0 \leqq \alpha < x < \infty$, our equations are q-difference-equations.) If a is an interior point of the interval I, we define the characteristic points of a function $t(x)$ with respect to a to be the zeros on the interval $a \leqq x < \beta$ (or $a \geqq x > \beta$) of the function $\bar{t}(x)$ derived from the constants $t(a)$, $t(S_a)$, $t(S_a^2)$, $\cdots$ by linear interpolation with respect to the system of coordinate axes obtained by drawing lines perpendicular to the x-axis through the points $a, S_a, S_a^2, \cdots$. Let u and v be real-valued, single-valued, continuous solutions of equations (45) and (46), respectively. Then we have the following three theorems:

If $u(x)$ has two consecutive characteristic points with respect to a on the μth and $(m+1)$th intervals $(\mu < m)$ of the set of intervals whose end-points are the consecutive pairs of the sequence $a, S_a, S_a^2, S_a^3, \cdots$, then $v(x)$ has a characteristic point between these characteristic points of $u(x)$ provided that either

(a) $\varphi(x) \leqq \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, the equality sign not holding for all the end-points of these intervals; or,

(b) $\varphi(x) = \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, and the two sets of constants

$$u(S_a^{\mu-1}),\ u(S_a^{\mu}),\ \cdots,\ u(S_a^{m-1}),$$
$$v(S_a^{\mu-1}),\ v(S_a^{\mu}),\ \cdots,\ v(S_a^{m-1}),$$

are linearly independent.

Next, let us suppose that $u(a) \neq 0$, $v(a) \neq 0$, $u(S_a)/u(a) > v(S_a)/v(a)$; and that $\varphi(x) \leqq \psi(x)$ for $x = a, S_a, S_a^2, \cdots, S_a^{\nu-1}$. If $u(x)$ has k characteristic points on the ν intervals whose end-points are the consecutive pairs of the sequence $a, S_a, S_a^2, \cdots, S_a^{\nu}$, then $v(x)$ has at least k characteristic points on these intervals; and the jth of these characteristic points of $v(x)$ (counted from a towards S_a^{ν}) is nearer to a than the jth characteristic point of $u(x)$.

In the third place, let $u(a)$, $v(a)$, $u(S_a^k)$, $v(S_a^k)$ be all different from zero and let $u(S_a)/u(a) > v(S_a)/v(a)$. Let $u(x)$ and $v(x)$ have the same number (which may be zero) of characteristic points on the k intervals whose end-points are the consecutive pairs of the sequence $a, S_a, S_a^2, \cdots, S_a^k$. Then we have

$$\frac{u(S_a^{k+1})}{u(S_a^k)} > \frac{v(S_a^{k+1})}{v(S_a^k)},$$

provided that $\varphi(x) \leqq \psi(x)$ for $x = a, S_a, \cdots, S_a^{k-1}$.

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HUYGENS' DIOPTRICA

Oeuvres complètes de Christiaan Huygens publiées par la Société Hollandaise des Sciences. Tome treizième. *Dioptrique.* 1653; 1666; 1685–1692. La Haye, Martinus Nijhoff, 1916. pp. i–clxviii; 1–905. 4to.

The monumental edition of the correspondence and writings of Christiaan Huygens was undertaken in 1888 by the Dutch Society of Sciences. The volume here under review, containing Huygens's original contributions to dioptrics and comprising with introduction, notes, etc., more than a thousand pages, is in every way worthy of its illustrious author. Unstinted praise should be bestowed on both the editor and the publisher, who have spared no pains in making this invaluable work as authoritative, complete and convenient as possible.

The manuscript of Huygens's writings on *Dioptrica* consists of detached sheets, 166 in all, the oldest portions of which date back to 1652, when Huygens was 23 years old. Almost until the end of his life in 1695, he was continually adding to it, revising it and sometimes planning to rewrite it entirely. By the end of 1653 he had composed a first draft of his *Tractatus de refractione et telescopiis* (pp. 1–271). At intervals during the following twelve years he was often at work on the same material, but for one reason or another its publication was constantly delayed. About 1665 he began to study more systematically the theory of spherical aberration; the results of these investigations are contained in the second part of the *Dioptrica* under the title of *De aberratione radiorum a foco* (pp. 273–353). About 1672, or perhaps a little later, we hear of his being again at work on his treatise on dioptrics with renewed ardour. By this time Newton's explanation of the phenomena of dispersion had been published. Its importance and its bearing on his own problems was quickly perceived by Huygens to the extent of modifying some of his views. Meanwhile, the undulatory theory had been born in his mind, and he was planning a more extensive treatise involving a revision and rearrangement of his previous manuscripts on dioptrics. In 1677 he found the explanation of double refraction in Iceland spar, which he naturally regarded as the most beautiful confirmation of his new theory of light. In comparison his earlier work in dioptrics seemed to him of secondary importance. Consequently, he determined to publish first a treatise on the wave theory of light with its principal applications, but without entering in detail into the theory of mirrors and lenses. This was the origin of his famous *Traité de la lumière.* Although it was not published until 1690, it was practically completed in 1678, and had been read before the Academy of Sciences in Paris in 1679. The third part of the *Dioptrica,* entitled *De telescopiis et microscopiis* (pp. 443–511), seems to have been composed in 1685. Apparently Huygens could never make up his mind to publish his optical theories and researches because he was continually adding fresh discoveries and gaining new insight and new points of view. In 1692 the work was still unfinished and at that time Huygens writes to Leibnitz: "il y a bien des choses à demesler dans cette Dioptrique, et il s'en est offert tousjours de nouvelles, jusqu'à cette heure, qu'il me semble d'avoir tout penetré, quoy que je n'aye pas encor achevé de tout escrire." And so, as a matter of fact, Huygens's *Dioptrica* was never actually completed, and was not published at all until after his death. The greater part of it was included in De Volder and Fullenius's edition of Huygens's *Opuscula posthuma* published in Leiden in 1703. Unfortunately, not a few of Huygens's most valuable theorems never saw the light until long after they were obtained. Consequently he lost the priority of a number of important discoveries.

In this volume the text of the *Dioptrica* is given in Latin with facsimile cuts of Huygens's original diagrams and illustrations. On the opposite page there is a French translation. In an introduction of 167 pages, the editor has reviewed the entire work in detail. The text itself is clarified and expounded by notes and comments. In short the reader has every aid that could be desired, including a complete and accurate index of the whole contents.

One of the chief fascinations in a lifelong work like the *Dioptrica* is to trace in it the voyages of a great and original mind. The reader is continually surprised not only by the impresses of genius which have been left on these pages, but perhaps most of all by the almost startling modernness of many of Huygens's conceptions, as was long ago remarked by the late Professor Silvanus P. Thompson, reminding us again and again how little in advance of men like Newton and Huygens we are even in the twentieth century. The outburst of optical science which followed the invention of the telescope, and which constitutes a kind of Elizabethan Era in the history of scientific discovery, seems, so to speak, to flash forth anew from between the covers of this volume, and we catch some spark of the joyous enthusiasm that possessed those eager men who first employed the telescope and the microscope. Somehow too from an occasional scornful allusion to the opinions of some of his predecessors or contemporaries, at least one reader has been led to suspect that Huygens had a human side also and did not suffer fools gladly!

Amid such a mass of material, it is difficult to single out one thing for special comment rather than another. Certainly it is worth noting with what elegance and skill—far in advance of his contemporaries in this respect—Huygens derives from the law of refraction the fundamental characteristics of optical imagery in the limiting case when the effective rays are nearly normal to the refracting surface. If these propositions had been published about 1653 when they were first obtained by Huygens, he would certainly have had the priority for them. They were communicated in an anagram to the Royal Society in 1669, but at that very time Dr. Isaac Barrow's *Lectiones opticæ* was in the press, in which were to be found essentially the same theorems derived in a different way. But Huygens had the idea of equivalent lenses which Barrow did not. When the *Dioptrica* was first published in 1703, other writers also, notably Molyneux in England, had given rules which were practically the same as those of Huygens.

Of much interest too is Huygens's way of defining and measuring the magnifying power of an optical instrument, by which he means the ratio of the apparent size of the object as seen through the instrument to its apparent size as presented to the unaided eye. Whether or not Huygens is entitled to priority for the announcement of the fact that the magnifying power of a telescope is equal to the ratio of the focal lengths of object-glass and ocular, he attached so much importance to it that he proves it in at least three different ways in the section on telescopes and microscopes which constitutes the third part of the *Dioptrica.* Neither Kepler nor Des Cartes had understood this relation clearly. In this connection speaking of Des Cartes, Huygens says that although it is hard to believe it of a man who was so intelligent and so well informed in these matters, nevertheless he got off the track in his demonstrations of the nature and effect of the telescope and was guilty of writing things on this subject to which no meaning can be attached!

A theorem which deserves to rank as one of the most beautiful generalizations of theoretical physics, and which can be extended to the theory of radiation in general, is contained in Proposition VI in the second book of the first part of Huygens's *Tractatus de refractione et telescopiis* (see pp. 198, ff.); it may be stated as follows:

"If an object is viewed through a system of any number of lenses, and if the positions of the eye and the object are mutually interchanged without disturbing the lens-system itself, the apparent size of the object will be the same as before, and the image will be erect or inverted as before."

This will be immediately recognized as equivalent to the theorem given by Robert Smith in his *Compleat System of Opticks* (Cambridge, 1738) as the first corollary to be deduced from Roger Cotes's celebrated proposition about the "apparent distance" of an object as viewed through a system of thin lenses. Smith does not mention Huygens's name in connection with this corollary, perhaps inadvertently; for undoubtedly at that time (1738) Smith must have known of Huygens's proposition. There can be no question that Huygens is entitled to the priority here. As early as 1653 Huygens, writing to Kinner von Löwenthurm, communicated this theorem as one of the principal discoveries which he proposed to publish in his treatise on refraction and telescopes; and in 1669 he included it in the anagrams which he sent to the Royal Society. The theorem itself was not actually published until 1703, but its importance was not appreciated and it was soon forgotten or ignored. One of the many services performed by the learned editor of this volume of Huygens's works is to put this matter in its right light, because the theorem in question is one, and by no means the least, of Huygens's titles to fame. Huygens himself constantly makes use of this general principle in the solution of special problems, and modern workers will find it serviceable in the same way.

Doubtless few persons are aware nowadays that Huygens made valuable contributions to science also in the realm of physiological optics, as may be seen by looking into this volume. Most physicists are content, so to speak, to deliver radiant energy to the eye and leave it to its fate; unfortunately, comparatively few of them like Young and Helmholtz have thought it worth while to pursue the investigation further and to study the intricate phenomena of vision. Not so Huygens; he at least was keenly alive to the fact that at the other end of his microscope or telescope a human eye was adjusted, and that visual perception is the chief thing after all. He was perfectly aware that the magnifying power of the instrument depended on the idiosyncrasies of the eye of the individual. Apparently far more accurately than Des Cartes or any of his own contemporaries, Huygens was acquainted with both the anatomical and optical structure of the eye; and he had the clearest notions about the office of the pupil and the mechanism of accommodation. The essential theory of binocular vision and depth perception was grasped by Huygens. He explains in the clearest manner how in order to see an object single with both eyes the two images on the retina must be formed at "corresponding points", although apparently he did not perceive that a solid object looks different to each eye; otherwise he might have anticipated Wheatstone and Brewster in the invention of the stereoscope. In the article on the eye and vision in the first part of the *Dioptrica* there is a description of a "simplified eye" formed by two concentric hemispheres of unequal radii. The curved surface of one of these hemispheres corresponds to the cornea and that of the other to the retina of the eye. There is a singular resemblance between Huygens's "simplified eye" and the "reduced eye" conceived by Listing in 1845, as is pointed out by the editor on page cxliv of the *Avertissement.*

After all, perhaps what impresses the reader most amid all this wealth of material is not so much the theories which are propounded and developed with such rare insight and skill, as the marvellous versatility and resourcefulness of the author and the variety of observations and experiments which underlie the whole and form the solid structure of the edifice itself. To his extraordinary mechanical ability and ingenuity Huygens owed much of his remarkable accomplishment; with him to conceive was to execute, no matter what practical obstacles might lie in the way. To this day we read with astonishment of those prodigious "aerial telescopes" with their poles and pulleys which he constructed and mounted with his own hands and with which he made some of his great discoveries in astronomy. Huygens's name is usually associated in our minds with refracting telescopes; but he devoted much study also to reflectors and preferred Newton's type of instrument to those of Gregory and Cassegrain. Many pages in the volume before us are devoted to the theory of the compound microscope and Huygens's "observations microscopiques."

But enough has been said to give the reader at least some idea of the character and scope of Huygens's optical researches. No wonder that he published comparatively little during his busy lifetime! Before he could get his thoughts safely on paper, a whole vista of new ideas begins to distract and fascinate him. New discoveries give ever a new turn to his earlier imaginings, and so he hastens onwards still eager in the pursuit of knowledge when death overtakes him at last at the summit of his great career.

Felix qui potuit rerum cognoscere causas.

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BOOKS ON RELATIVITY

Das Relativitätsprinzip. Lorentz. Einstein. Minkowski. Fortschritte der mathematischen Wissenschaften in Monographien. Herausgegeben von O. Blumenthal, No. 2. Leipzig und Berlin, B. G. Teubner, dritte Auflage, 1920. i + 146 pp.

Raum. Zeit. Materie. Von Hermann Weyl. Berlin, Julius Springer, vierte Auflage, 1921. Mit 15 Textfiguren. ix + 300 pp.

Relativity. The special and the general Theory. By Albert Einstein. Translated by Robert W. Lawson. New York, Henry Holt and Co., 1921. Frontispiece. xiii + 168 pp.

The Theory of Relativity. By Robert D. Carmichael. Mathematical Monographs, Edited by Mansfield Merriman and Robert S. Woodward, No. 12. New York, John Wiley and Sons, 2nd edition, 1920. 112 pp.

Das Relativitätsprinzip. Leichtfasslich entwickelt von Adam Angerbach. Leipzig und Berlin, B. G. Teubner, 1920. Mit 9 Figuren im Text. 57 pp.

The Concept of Nature. Tarner Lectures delivered in Trinity College, November, 1919. By A. N. Whitehead. Cambridge, The University Press, 1920. viii + 202 pp.

Wiskunde, Waarheid, Werkelijkheid. Door L. E. J. Brouwer. Groningen, P. Noordhoff, 1919. 12 pp. + 23 pp. + 29 pp.

For scientists generally, and especially for mathematicians and physicists, who understand best many of the questions involved, the theory of relativity has fundamental interest. In the following pages our purpose is to pass in review the above recent books dealing with the theory and at the same time to indicate its present state and some unsolved problems.

The collection of monographs gathered by Blumenthal begins with two papers by the Dutch physicist, Lorentz, the second and more important one of which appeared in 1904. By endeavoring to unite the classical Newtonian mechanics and the electromagnetic theory of Faraday and Maxwell into a single consistent theory, one is necessarily led to absolute space (the ether) and absolute time. In fact, physics has stood committed to absolute time since the acceptance of Newton's law of gravitation. But the experiments of Michelson in 1881 yielded an opposing result. Lorentz, in common with other physicists, had the conviction that the universe was electromagnetic in character, and he turned to the electromagnetic equations for an explanation of the difficulty. His answer to the apparent contradiction of theory and experiment was based upon the fact that the equations admitted of a transformation in which space and time were intermingled. On this basis, without giving up the concepts of absolute space and time, he was able to explain the paradox by assuming that bodies undergo a slight contraction in the direction of their motion, which for the earth is not more than a few inches. To an observer moving at uniform velocity, the same electromagnetic equations appear to hold because such an observer uses "local time." Lorentz's explanation violated a fundamental principle, namely the pragmatic principle that no physical entity exists if its presence can never be determined by any conceivable experiment. Absolute space and time are this type of entity in his theory.

A year later in 1905, but independently, Einstein wrote his paper *Zur Elektrodynamik bewegter Körper,* which is the third paper of the collection. In this he lays the foundation of the so-called special theory of relativity. Einstein starts with a peculiarly simple type of physical universe, perhaps the simplest in harmony with all known physical laws. This is the universe of empty isotropic space in which there are infinitesimal inertia particles. The particles *appear* from such a particle to move with uniform velocity in a straight line, if observations of light signals are made with the aid of a clock. Thus the fundamental measuring instrument is the clock. It is further assumed that a light pulse appears to advance at a constant velocity from any such particle (the Michelson experiment). On the basis of these postulates the transformation equations between the coordinates set up from reference particles are deduced, and it is shown that the Maxwell electromagnetic equations are unaltered under precisely this group (the Lorentz group) of transformations. The behavior of the electron as experimentally determined is in conformity with this theory. In the short paper that follows Einstein notes that the same discussion indicates that the apparent mass of a system will depend upon its energy.

The fifth article of the collection is the mathematician Minkowski's remarkable *Raum und Zeit* of 1908. In this article, which threw a flood of light upon the work of Einstein and Lorentz, the geometry of four dimensions furnished the principal weapon. If Minkowski had lived, doubtless other equally important contributions to the theory of relativity would have come from his pen, and in any case it is clear that his influence upon Einstein can scarcely be overestimated.

The gist of Minkowski's paper is as follows: In the four-dimensional relativistic manifold of space and time, a pair of world-points or events are associated with a unique number, namely, if a particle move from the earlier world-point to the later world-point, the interval of local time elapsed will give this number. The mathematician will realize at once that we have here the elements of a non-euclidean geometry of four dimensions of simple type. The Lorentz transformations are merely the transformations of the geometry which leave this interval between world-points unaltered, and the whole theory may be subsumed in the single equation

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

where ds is the local time element, c is the velocity of light, and dx, dy, dz, dt have their customary meanings. In the same article it is pointed out how the laws of motion of the electron may be interpreted in this space; and a suitable modification of the Newtonian law of attraction is made, such as Poincaré had given earlier. There follow some instructive notes by the mathematical physicist Sommerfeld.

The remaining articles of the collection are reprints of more recent articles by Einstein, and four of them do not appear in the earlier editions. It is *Die Grundlagen der allgemeinen Relativitätstheorie* of 1916 which has aroused such widespread attention. Concerning it, Sommerfeld says in the notes just mentioned: "This general relativity theory is logically so unified and satisfactory that it has found unconditional acceptance, especially in mathematical quarters." In a paragraph added after learning of the verification of Einstein's quantitative prediction of the deviation of light by the sun, Sommerfeld says further, "The general relativity theory can therefore be regarded as an *established proposition.*" If this is the truth, physical science is entering upon an era in which the new view will differ radically from the classical one.

To the mathematician, Einstein's generalized theory is of interest in several respects. In the first place it illustrates afresh the importance of taking the simplest possible case as an abstract basis of departure. Secondly, Einstein uses mathematical analogy in passing, step by step, from the simple universe of the special theory to the most general universe, and at each step the mere sense of mathematical form is sufficient to point toward a natural generalization. The mathematician may feel satisfied that the formal analogies supplied by classical dynamics and four-dimensional geometry furnish the very basis by which Einstein's generalization proceeds. And, thirdly, the technical tool which made elaboration and verification of the theory possible is the invention of the mathematicians Riemann, Christoffel, and more especially of Ricci and Levi-Civita—namely the absolute differential calculus.

What then are these successive steps of Einstein? To the writer they appear as follows:

(1) In the special theory of relativity, the universe consists of an empty isotropic space with infinitesimal inertia particles, and the central formula is that for ds^2 given above.

(2) A somewhat more general type of universe is that of a non-isotropic empty space formed by a gravitational field. It is natural, by analogy with the Riemann geometry, to assume that local time is given by a quadratic differential form ds^2 in the space and time variables, and that the particular coordinates chosen are irrelevant (general theory of relativity). But in such case only six of the ten coefficients in ds^2 must be regarded as arbitrary. Therefore there are required six equations to fix these coefficients and these conditions must be independent of the coordinate system. This leads to Einstein's conclusion that the contracted Riemann tensor vanishes. By analogy with the special theory of relativity, the paths of the particles appear as the geodesics and the paths of the light pulse satisfy the equation $ds = 0$.

(3) Still more generality is obtained if matter and energy are present. For case (1), this leads to the vanishing of the divergence of an "energy tensor." In the equations obtained in case (2) the left-hand members are tensors while the right-hand members vanish. It is natural then to assume by analogy with classical dynamics that, in case the space contains matter and energy, the right-hand member becomes the energy tensor. If we

assume this to be the case, the complete equations, as general in their scope as those of the classical theory, are obtained.

The causal principle is effective, but in an obscure form, as follows: At any "instant" for the coordinate system under consideration the time rate of change of the derivatives of the gravitational tensor formed by the system of coefficients in ds^2 and of the energy tensor are known. Also the apparent accelerations of the particles are thereby determined. Consequently it is possible to obtain the new value of the tensors and their derivatives, and the new positions and velocities of the particles, an instant later, and so to proceed indefinitely.

According to Einstein the acceleration of a particle in empty space is due to gravitational forces, and depends only on ds^2 and the coordinates chosen. The law of motion at low velocities is found to be the same as that of Newton except for very small modifications. To make a specific application, Einstein determines the necessary form of ds^2 for a single central body such as the sun, also on a postulational basis, and arrives at his brilliant predictions.

In the final paper of the collection, written in 1919, Einstein shows how the idea of a spherical space may be conveniently introduced to eliminate the difficulties due to boundary conditions.

The casual reader of the theory of relativity will feel a certain lack of concreteness. The classical physical theories seemed to touch reality in at least three ways, namely in the independent concepts of space, of time, of force. I take it to be self-evident that any genuine physical theory must touch reality somewhere. So far as I can see, the Einstein theory does this at one and at only one place, namely in its concept of local time which can be measured by means of the natural clock, the atom. In an article appearing in January, 1921, in the PROCEEDINGS OF THE BERLIN ACADEMY, Einstein lays emphasis upon this notion of the natural clock as the fundamental element, but one could wish that this had been done more definitely in his earlier articles.

From the scientific point of view the most important of the other books which we desire to review is Weyl's *Raum. Zeit. Materie.* The Einstein theory has a certain pliability in the presence of an energy tensor, which may be modified to suit the exigencies of the physical situation under discussion. On the other hand, this pliability will appear as a defect to some minds since it provides physical science with a blank form rather than with a definitive theory. It may naturally be expected that theories will be forthcoming which attempt to explain non-gravitational phenomena also on a similar quasi-geometrical basis. One recalls here the vortex theory of the atom as an analogous attempt in classical physics.

The original part of Weyl's valuable and complete treatise consists in an attempt to deduce the electromagnetic equations in such a manner. For this purpose he invents a generalization of the Riemann geometry. In the Riemann geometry, the elements ds^2 can be compared at various parts of the manifold. Weyl notes that this is a species of action at a distance and proposes to compare the elements ds^2 only for various directions at a world-point. In other words his quadratic form ds^2 is one in which merely the ratios of the coefficients are important and the coefficients appear as undetermined up to an arbitrary multiplicative scale-factor.*

Thus in the normalization of his quadratic form by change of variables, he has five arbitrary functions (the four arbitrary coordinate functions and his scale-function) instead of the four functions available in the Einstein theory. Weyl is able to use the notion of parallel displacement due to Levi-Civita; namely, the small vector can be displaced so as to maintain size and direction in a specially chosen *geodesic* coordinate system. As this vector varies in position and returns to its starting point it will not have the same length except in the case of the Riemann geometry. The logarithmic derivative of the scale-function is a differential, whose four components behave as the components of an electromagnetic potential.

An obvious objection to Weyl's extension is that he loses contact with the real, for ds can no longer stand for the element of local time; otherwise we should expect atoms of the same element with different past histories to have different rates of vibration, and such has never been observed to be the case. In the second place, a modified ds^2 for the same manifold can be obtained which is invariant as in the Riemann geometry, and thus we are led back to the Einstein theory together with a single independent equation of the type coming under Einstein's theory. These criticisms of Weyl's work are given in the March, 1921, number of the PROCEEDINGS OF THE BERLIN ACADEMY by Einstein. Eddington has proposed a further modification of Weyl's theory in the April number of the PROCEEDINGS OF THE ROYAL SOCIETY. Eisenhart and Veblen have gone much further in an important paper in the PROCEEDINGS OF THE NATIONAL ACADEMY, February, 1922.

We pass now to the more popular treatments given in the next three books. The first of these is by Einstein himself, and affords an interesting and skillful approach to the fundamentals of the theory.

The second edition of Carmichael's book contains his earlier treatment of the special theory based upon a set of physical postulates. The new chapters are a direct summary of the results of the general theory, as presented by Einstein and Eddington. This summary is too abbreviated to be followed with much profit by the reader who has not delved elsewhere into the theory.

The tiny pamphlet by Angersbach presents a brief historical development of the notions underlying the special relativity together with an elementary presentation. It is readable.

It is obvious that the relativity theory has decided significance for philosophical thought, as indeed every new physical theory must have. In his interesting book, *The Concept of Nature*, the English philosopher-mathematician Whitehead expounds his views of the physical universe in the light of the theory of relativity. The book has obvious relations with an earlier book. † The main idea of Whitehead is that the underlying realities are attained by a method of extensive abstraction, a spatial point for instance being generated by all the objects of a certain category (those which include it spatially). His analysis of experience is very interesting. The mathematician will regret frequently redundancy and vagueness in philosophical treatises. Of this there is little in Whitehead's book. The importance and the exactitude of many of his analyses must be admitted. Maxime Bôcher once said to the writer, "What man would be a philosopher who might be a mathematician!" One feels that Mr. Whitehead deserves both titles.

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* See his recent papers in the MATHEMATISCHE ZEITSCHRIFT, vol. 12, Nos. 1, 2 (1922).

† *An Inquiry Concerning the Principles of Natural Knowledge*, A. N. Whitehead. Cambridge, University Press, 1919. xii + 200 pp.

The contrasting account given in this book between the old theories and the new theory of relativity is interesting. Characterizing the old theories, Whitehead says: "For example, colour is the result of a transmission from the material object to the perceiver's eye; and what is thus transmitted is not colour. Thus colour is not part of the reality of the material object. Similarly for the same reason sounds evaporate from nature. Also warmth is due to the transfer of something which is not temperature. Thus we are left with spatio-temporal positions, and what I may term the 'pushiness' of the body. This leads us to eighteenth and nineteenth century materialism, namely, the belief that what is real in nature is matter, in time and in space and with inertia."

The new relativity theory he expounds as follows: "Let us make therefore the general statement that four measurements, respectively of independent types (such as measurements of lengths in three directions and a time) can be found such that a definite event-particle is determined by them in its relations to other parts of the manifold. . . . If (p_1, p_2, p_3, p_4) be a set of measurements of this system, then the event-particle which is thus determined will be said to have p_1, p_2, p_3, p_4 as its coordinates in this system of measurement." . . . "Then we should naturally say that (p_1, p_2, p_3) determined a point in space and that the event particle happened at that point at the time p_4. . . . Furthermore the inhabitant of Mars determines event-particles by another system of measurements. Call his system the q-system. According to him, (q_1, q_2, q_3, q_4) determines an event-particle, and (q_1, q_2, q_3) determines a point and q_4 a time. But the collection of event-particles which he thinks of as a point is entirely different from any such collection which the man on earth thinks of as a point. Thus the q-space for the man on Mars is quite different from the p-space for the land-surveyor on earth. . . ."

". . . We have got to find the way of expressing the field of activity of events in the neighborhood of some definite event-particle E of the four-dimensional manifold. I bring in a fundamental physical idea which I call the 'impetus' to express this physical field. The event-particle E is related to any other neighboring event-particle P, by an element of impetus." ··· "Einstein showed how to express the characters of the assemblage of elements of impetus of the field surrounding an event-particle E in terms of ten quantities which I will call $J_{11}, J_{12}, \ldots$ The numerical values of the J's will depend on the system of measurement adopted, but are so adjusted to each particular system that the same value is obtained for the element of impetus between E and P, whatever be the system of measurement adopted. This fact is expressed by saying that the ten J's form a 'tensor.' " . . . "We now return to the path of the attracted particle. We add up all the elements of impetus in the whole path, and obtain thereby what I call the 'integral impetus.' The characteristic of the actual path as compared with neighbouring alternative paths is that in the actual paths the integral impetus would neither gain nor lose, if the particle wobbled out of it into a small extremely near alternative path."

Evidently Whitehead is expressing the relativistic theory of the path of a particle in a gravitational field. The indefinite general terms used stand in unfavorable contrast with those which can be used in the exposition of the classical theory.

Finally we turn to the little pamphlet by Brouwer. Many American mathematicians have read in this BULLETIN (November, 1913) a translation of Brouwer's paper on *Intuitionism and Formalism*, which is the final essay of the pamphlet. Those who know the mathematical work of Brouwer will be interested in these essays, in the second of which he touches upon the special theory of relativity with emphasis upon the notion of *group*. Attention should also be directed to his noteworthy analysis of the logical principle of the *excluded middle*, to which reference is made in the first essay.

In conclusion, one or two remarks of general character suggest themselves.

The theory of relativity in its general form or in its special form involves a definite group. To the mathematician at least it would be of considerable interest to see physical theories developed for other special groups. In particular the most general group of all, that of *analysis situs*, suggests itself, for this alone appears strictly proper in the general theory of relativity, where any transformation of coordinates whatsoever ought to be admitted. If such a theory can be constructed, the interrelation of continuous manifolds of arbitrary form will form the essential element. A theory of this kind would seem to be consonant with quantum theory.

Also, with others, we may call attention to the fact that no theory of relativity so far explains the difference between positive and negative electricity, or throws any light upon the constitution of matter. Furthermore no real reason appears why the velocities of the stars relative to one another are so small in comparison with the velocity of light.

While awaiting further developments, let us at least say with Whitehead of Einstein's investigations: "They have made us think." Some may agree with the final words of Weyl: "A few chords of that harmony of the spheres of which Pythagoras and Kepler dreamed have fallen upon our ears."

G. D. BIRKHOFF.

SHORTER NOTICES

Einführung in die theoretische Physik. Von Clemens Schaefer. Band II, Teil 1. *Theorie der Wärme, Molekular-kinetische Theorie der Materie.* Berlin, Vereinigung wissenschaftlicher Verleger, 1921. x + 562 pp.

The first volume on mechanics (particles, rigid bodies and continua) when reviewed in these pages was signalized for its excellence of material and of style. The second volume continues the excellences of the first. The rest of the work will be awaited with interest, and, by those engaged in teaching theoretical physics, with impatience. It is a pity we cannot have such books in English. Twenty years ago there was hope that Andrew Gray's *Treatise on Physics* (of which vol. I, *Dynamics*, alone appeared) would fill this lacuna. Schaefer has written his present volume since the Armistice. He has been at Breslau and is now at Marburg.

There are eleven chapters as follows: 1. Heat conduction. 2. First Law of thermodynamics. 3. Second Law of thermodynamics. 4. Homogeneous systems. 5. Heterogeneous systems. 6. Special systems (gases and dilute solutions). 7. Chemical affinity and Nernst's theorem. 8. Kinetic theory of gases. 9. Entropy and probability. 10. Statistical mechanics. 11. Quantum theory.

The arrangement is logical. General old-fashioned theories of heat conduction and thermodynamics come first—general thermodynamics, be it noted, not the special idealized systems. The treatment of chemical affinity is more modern but at least in its main lines is now firmly established. The last four statistical chapters are excellent: no over-elaboration on ancient kinetic theory, a clear treatment of the fundamentals of statistical mechanics, and an excellent and convincing presentation of quantum theory so far as heat phenomena are concerned. Nothing is said of radiation—not even the Stefan law which sometimes is deduced from a Carnot cycle and could have been inserted in Chapter III as illustrative material.

The last few pages on Nernst's theory of the degeneration of ideal gases (Entartung) should have been omitted in a book of this kind. It is too special and not yet well enough substantiated either experimentally or theoretically (as the author notes). The space thus gained could well have been given to real gases, under which only van der Waals's equation is given, whereas for some not remote purposes other equations are useful; and for accurate descriptions of the behavior of substances over long ranges of temperature, pressure, and density, Keyes's equation is incomparably superior to van der Waals's.

Generally the mathematical treatment is elegant—not the elegance of the mathematical virtuoso, but that of the working physicist. (A notable exception is the painful summation, p. 502, of the series Σnx^n in place of its evaluation as the derivation of Σx^n multiplied by x.) In the main, physical concepts are kept well in the foreground throughout mathematical discussions, and physical philosophy is delightfully in evidence throughout involved physical discussions. Tables of numerical data help at times to increase the effective reality of the work. One thing, however, continental authors disregard to their serious detriment, and that is exercises for the student. When exercises are well selected they double the educational value of any book. Schaefer has the teacher's instincts and could easily have furnished excellent exercises.

It would be only too easy to stretch this review to great lengths, mentioning such clear-cut analysis as that of Boltzmann's "ergodische" systems with its unusual frankness in pointing out that there are no such systems (p. 439), of the total separation of Liouville's theorem and the equipartition of energy, etc., of the discussion of anomalies (Schwankungen) and its bearing on the work of Perrin and his followers. But there are too many details to cite, and we had best not begin. Let the book be widely read.

EDWIN B. WILSON.

The Elements of Non-Euclidean Geometry. By D. M. Y. Sommerville. Chicago, The Open Court Publishing Company; and London, G. Bell and Sons, 1919.* xvi + 274 pp.

The printing of a second edition of this book, as well as the fact that it now, for the first time, appears as a publication of the Open Court Publishing Company, speaks well for the attention it has received from mathematical readers.

A careful comparison of the two editions shows no changes. The very few typographical errors and the somewhat more frequent unintended slips in the text remain. For instance, on p. 10, l. 20, a parenthesis is still lacking; likewise an "*s*" on p. 22, l. 1. There remains (p. 54) this remarkable series of statements:—"A triangle has therefore four circumcircles. ··· There cannot be more than one real circumcentre" [and may, of course, be none]. "This point, which we may call *the* circumcentre, ··· may be real, at infinity, or ideal." On p. 204, line 19, *R* is still called the earth's radius, although the radius of the earth's orbit is clearly meant.

It is to be sincerely hoped that the popularity of the book may before long require still another printing, and that a careful revision may then make it as accurate as it is interesting.

EDWARD S. ALLEN.

Annuaire du Bureau des Longitudes pour 1921. Publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1921. 7 + 710 + 130 pp.

Before 1904, each issue of the Annuaire contained all the information which the Bureau considered necessary for publication. Owing to the number of pages, which had gradually increased, it was then divided and certain articles in which there was little or no annual change were given only in alternate years. The volume has now grown again to a size which is near its maximum for convenient handling and one finds references which include the previous four volumes. Its chief annual feature is the full astronomical information for the current year and a briefer summary for the succeeding year. There are usually one or more appendices containing articles on various topics. This year M. Bigourdan writes on proper motions and radial velocities of stars—a summary which, starting at the beginning, takes the reader to the work which is now in progress at many observatories. A valuable feature is the full index of all matters contained in the current issue with references, where necessary, to the previous four volumes.

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* Originally published in 1914 by G. Bell and Sons, and reviewed in this BULLETIN, vol. 21, May, 1915, by J. L. Coolidge.

Ernest W. Brown.

Die Grundlagen der Geometrie als Unterbau für die Analytische Geometrie. By Lothar Heffter. Leipzig and Berlin, B. G. Teubner, 1921. ii + 27 + viii pp.

The author's purpose is to provide a foundation for the study of analytic geometry and this he proposes to do by stating a set of projective postulates sufficient for the introduction of the double ratio, the fundamental concept of the projective scale. He explicitly states his intention to ignore the question of the independence of his postulates—and would have us regard them as "one possible way of idealizing the results of observation" rather than as a purely abstract gift from heaven. His undefined elements are point, line, plane and incidence. He lists twenty "axioms of alignment" (Verknüpfung), arranged in ten pairs of dual propositions, three "axioms of order" (separation), and one of continuity. To secure the restriction to the affine and then to the euclidean metric geometries, he introduces one axiom of parallelism and one axiom of orthogonality. To those interested in a brief formulation of the postulational foundation, from a projective point of view, for the study of analytic geometry the little pamphlet will be of value.

J. W. Young.

Les Théories d'Einstein. By Lucien Fabre. Paris, Payot et Cie., 1921. 225 pp.

This book will be found of interest because of its valuable survey of the historical background of Einstein's theories. The author sometimes fails to define the symbols used. (For example, on pages 161, 179, and 243, and on page 196 there are incorrect definitions.) In spite of these defects, however, the book is one of the best elementary expositions of Einstein's work that has been published.

C. N. Reynolds.

NOTES

At the meeting of the Division of Physical Sciences of the National Research Council in Washington on April 23, the following officers were elected for the year beginning June, 1922: Chairman, William Duane; Secretary, W. E. Tisdale; Executive Committee, G. D. Birkhoff, William Bowie, C. E. Mendenhall, A. Trowbridge. The following persons were elected members-at-large: G. A. Bliss, F. A. Saunders, A. Trowbridge. New members representing scientific societies were announced as follows: for the American Astronomical Society, E. B. Frost; for the American Mathematical Society, C. N. Haskins; for the American Physical Society, H. G. Gale and C. E. Mendenhall.

The annual meeting of the National Council of Mathematics Teachers was held at Chicago on March 1, 1922, in connection with the meeting of the Department of Superintendence of the National Education Association. A number of papers were read, dealing with the teaching of secondary mathematics.

Professor W. F. Osgood and G. D. Birkhoff have been elected corresponding members of the Göttingen Academy of Sciences.

Professor M. Planck has been elected a foreign member of the Swedish Academy of Sciences.

Professor A. Gutzmer has been elected president of the Leopoldinisch-Carolinische Akademie der Naturforscher, as successor to Professor A. Wangerin, who has resigned. Professors H. Hahn and R. Schüssler have been elected members of this Academy.

Professor E. I. Fredholm, of the University of Stockholm, has been elected correspondent of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor H. A. Schwarz.

Requests to authors of mathematical works for reprints have been sent by L'Istituto Matematico della R. Università di Bologna.

Cambridge University has awarded a Rayleigh prize to T. A. Brown, of Trinity College, for an essay *On a class of factorial series*.

The honorary degree of doctor of science has been conferred on Sir Thomas Muir by the University of Cape Town, in recognition of his researches in mathematics and the history of mathematics.

Professor A. S. Eddington has been elected president of the Royal Astronomical Society.

In connection with the announcement of the distribution of reprints of articles by the late Professor Maxime Bôcher (this BULLETIN, vol. 28

(1922), p. 74), it is stated that the reprints available, numbered as in this Bulletin, vol. 25 (1919), p. 209, are as follows: 1, 3, 4, 6, 7, 9, 12, 16, 17, 18, 21, 23, 25, 26, 30–34, 36–42, 48, 51, 52, 53, 56, 59, 60, 63, 64, 66–69, 74, 78, 79, 83, 84, 85, 87–92, 97, 99, 100. Reprints will be sent, while the supply lasts, to those applying by number for the ones desired to Professor O. D. Kellogg, 20 Craigie St., Cambridge 38, Mass. The recipient will be expected to defray mailing costs; no other charge will be made except for No. 67, *Introduction to the Theory of Fourier Series* ($0.75) and No. 99, *Leçons sur les Méthodes de Sturm* ($1.25).

Professor G. D. Birkhoff will deliver four lectures on the theory of relativity, and four lectures on the problem of three bodies, at Stanford University during the latter part of June, 1922.

Dr. H. Brandt, of the Karlsruhe Technical School, has been appointed to a professorship at the Technical School at Aachen.

Dr. W. Schmeidler, of the University of Kiel, has been appointed to a professorship at the Breslau Technical School.

Dr. O. Szàsz has been promoted to an associate professorship at the University of Frankfurt a. M.

Dr. F. Bernstein has been promoted to a full professorship at the University of Göttingen.

Professor J. Radon, of the University of Hamburg, has been appointed full professor of mathematics at the University of Greifswald.

Dr. G. Prange, of the University of Halle, has been appointed to a full professorship at the Hannover Technical School.

Professor F. Noether has been appointed associate professor of applied mathematics at the University of Heidelberg.

Dr. Pohlhausen has been promoted to an associate professorship of applied mathematics at the University of Rostock.

At the Stuttgart Technical School, Dr. K. Kommerell has been promoted to an associate professorship, and Dr. F. Pfeiffer, of the University of Heidelberg, has been appointed full professor.

Dr. R. König has been promoted to a full professorship of mathematics at the University of Tübingen.

In the Faculty of Sciences of the University of Paris, Professor P. Montel has been appointed to the professorship of general mathematics, and Professor Leduc to the professorship of theoretical physics, as successors, respectively, to Professors J. Drach and E. Cotton, called to other professorships in the same faculty. The title of the chair of "differential and integral calculus" has been changed to "the theory of groups and the calculus of variations."

Mr. A. L. Dixon, fellow and tutor of Merton College, has been appointed Wayneflete professor of pure mathematics at Oxford University, as successor to Professor E. B. Elliott.

Dr. G. B. Jeffery has been appointed to the University chair of mathematics at King's College, London University.

Professor W. F. Osgood has been appointed acting dean of the graduate school of arts and sciences at Harvard University for the second half of the present academic year.

At the University of Rochester, Professor A. S. Gale has been appointed Freshman Dean.

At Yale University, Assistant Professor Leigh Page, of the department of physics, has been promoted to a full professorship of the mathematical sciences, with assignment to the Sheffield Scientific School. Dr. W. L. Crum has been promoted to an assistant professorship of mathematics. Professor L. S. Hill, of the University of Maine, has been appointed instructor in mathematics.

Dr. E. B. Wilson, professor of mathematical physics at the Massachusetts Institute of Technology, has been appointed professor of vital statistics at Harvard University.

Assistant Professor C. G. P. Kuschke has been promoted to be full professor and head of the department of mathematics at the University of Porto Rico.

At the University of Kansas, for the year 1920–21, Dr. U. G. Mitchell was advanced to a full professorship and Dr. Guy Smith was appointed assistant professor; for the year 1921–22, Dr. E. B. Stouffer was advanced to a full professorship.

Dr. H. C. M. Morse and Dr. W. L. G. Williams have been promoted to assistant professorships at Cornell University.

Professor G. Kohn, of the University of Vienna, died December 15, 1921, at the age of sixty-two years.

Professor E. Jahnke, of the Berlin Technical School, died October 18, 1921, at the age of fifty-eight years.

Professor L. Koenigsberger, of the University of Heidelberg, died December 15, 1921, at the age of eighty-three years.

Professor M. Noether, of the University of Erlangen, died December 13, 1921, at the age of seventy-seven years.

Dr. G. B. Mathews, formerly professor of mathematics at the University College of North Wales, died March 19, 1922, at the age of sixty-one years.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

ARCHIMEDES. See VER EECKE (P.).

ARWIN (A.). Die Kongruenzen $(\lambda + 1)^p \equiv 0 \pmod{p^\nu}$, und die Natur ihrer Lösungen. Leipzig, 1921. 4to. 38 pp.

BAKER (H. F.). Principles of geometry. Volume 1: Foundations. Cambridge, University Press, 1922. 8vo. 12 + 184 pp.

BLASCHKE (W.). Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. I: Elementare Differentialgeometrie. (Die Grundlehren der mathematischen Wissenschaften, herausgegeben von R. Courant, Band 1.) Berlin, Springer, 1921. 10 + 230 pp.

BOREL (E.). Leçons sur les fonctions entières. 2e édition. Paris, Gauthier-Villars, 1921. 9 + 161 pp.

CONGRÈS INTERNATIONAL. See VILLAT (H.).

COURANT (R.). See BLASCHKE (W.), KNOPP (K.).

DE MORGAN (A.). A budget of paradoxes. Edited by D. E. Smith. 2d edition. Chicago and London, Open Court, 1921. 2 volumes.

DEUTSCHBEIN (M.). Wahrscheinlichkeit und Induktion. (Diss., Halle.) Cöthen, Verlag von Otto Schulze, 1920.

EHRENFELS (C.). Das Primzahlengesetz entwickelt und dargestellt auf Grund der Gestalttheorie. Leipzig, Reisland, 1922.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band II 3, Heft 5–6: K. Hensel, Arithmetische Theorie der algebraischen Funktionen. H. W. E. Jung, Arithmetische Theorie der algebraischen Funktionen zweier unabhängigen Veränderlichen. Leipzig, Teubner, 1921.

EULER (L.). Opera omnia. Series I: Opera mathematica. Volumen VI: Commentationes algebraicae ad theoriam aequationum pertinentes. Ediderunt F. Rudio, A. Krazer, P. Stäckel. Leipzig, Teubner, 1921. 20 + 509 pp.

FISHER (A.). The mathematical theory of probabilities. Volume 1. Reissue, with new chapters. New York, Macmillan, 1922.

FRICKE (R.). Die elliptischen Funktionen und ihre Anwendungen. Teil 2: Die algebraischen Ausführungen. Leipzig, Teubner, 1922. 8vo. 8 + 546 pp.

GRÜNBERG (J.). Die "D"-Linien der Mittelpunktsflächen zweiten Grades. (Diss., Heidelberg.) Göttingen, Dietrichsche Universitäts-Buchdruckerei, 1920.

HEATH (T.). A history of Greek mathematics. Volume 1: From Thales to Euclid. Volume 2: From Aristarchus to Diophantus. Oxford, Clarendon Press, 1921. 15 + 446 + 11 + 586 pp.

VON HELMHOLTZ (H.). Schriften zur Erkenntnistheorie. Herausgegeben und erläutert von P. Hertz und M. Schlick. Berlin, Springer, 1921. 10 + 176 pp.

HENSEL (K.). See ENCYKLOPÄDIE.

HERTZ (P.). See VON HELMHOLTZ (H.).

HUMBERT (P.). Introduction à l'étude des fonctions elliptiques. Paris, Hermann, 1922. 8vo.

JOHNSON (W. E.). Logic. Part 1. Cambridge, University Press, 1921. 40 + 255 pp. 15s.

JUNG (H. W. E.). See ENCYKLOPÄDIE.

KEYSER (C. J.). Mathematical philosophy. A study of fate and freedom. Lectures for educated laymen. New York, Dutton, 1922. 14 + 466 pp. $4.70

KLAUCKE (F.). Ueber Raumkurven, zwischen deren beiden Krümmungen eine Beziehung besteht. (Diss., Halle.) Halle, H. John, 1916. 128 pp.

KNOPP (K.). Theorie und Anwendung der unendlichen Reihen. (Die Grundlehren der mathematischen Wissenschaften, herausgegeben von R. Courant, Band 2.) Berlin, Springer, 1922. 10 + 474 pp.

KOSCHMIEDER (L.). Untersuchungen über Jacobischen Polynome. Habilitationsschrift. Breslau, Trewendt und Granier, 1919.

KRAZER (A.). See EULER (L.).

LINDEMANN (F.). Die Biegungsflächen einer gegebenen Fläche. München, 1921.

LOBATTO (—.). See SCHUH (F.).

MALET (H.). Etude géométrique des transformations birationnelles et des courbes planes. Paris, Gauthier-Villars, 1921. 8vo. 262 pp.

VON MANGOLDT (H.). Einführung in die höhere Mathematik für Studierende und zum Selbstudium. 1ter und 2ter Band, 3te Auflage. 3ter Band, 2te Auflage. Leipzig, 1921.

MARCHAND BEY (E. E.). Mathématiques inédites. Etude complète des spirales logarithmiques par nos méthodes élémentaire et analytique. Epiry, chez l'auteur, et Paris, Desforges, 1921. 8vo. 87 pp.

NEVILLE (E. H.). The fourth dimension. Cambridge, University Press, 1921. 8vo. 56 pp.

PASCAL (E.). Repertorium der höheren Mathematik. 2ter Band, 2te Hälfte: Raumgeometrie. Zweite völlig umgearbeitete Auflage der Deutschen Ausgabe, herausgegeben von H. E. Timmerding. Leipzig, Teubner, 1922. 12 + 629 pp.

PLATEN (C.). Beiträge zur Untersuchung der harmonischen Kovarianten zweier Kurven. (Diss., Halle.) Halle, Hohmann, 1920. 60 pp.

ROUGIER (L.). Les paralogismes du rationalisme; essai sur la théorie de la connaissance. Paris, Alcan, 1921. 8vo.

RUDIO (F.). See EULER (L.).

RUSSELL (B.). The analysis of mind. New York, Macmillan, 1921. 310 pp.

SCHILLING (B.). Ueber eine Berührungstransformation, die den Punkten des einen Feldes Geradenpaare zuordnet. (Diss., Dresden.) Borna-Leipzig, Druck von R. Noske, 1919.

SCHLICK (M.). See VON HELMHOLTZ (H.).

SCHUH (F.). Lessen over de hoogere algebra. Eerste deel. Uitgegeven als negende druk van Lobatto's Lessen over hoogere algebra. Groningen, Noordhoff, 1921. 20 + 526 pp.

SMITH (D. E.). See DE MORGAN (A.).

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THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The two hundred twenty-first regular meeting of the Society was held at Columbia University on Saturday, February 25, extending through the usual morning and afternoon sessions. The attendance included the following seventy-five members:

Alexander, Beal, Birkhoff, Borden, Bowden, E. W. Brown, T. H. Brown, E. T. Browne, B. H. Camp, G. A. Campbell, Cole, Coolidge, Cowley, Cronin, Dantzig, Douglas, Eisenhart, Fine, Fischer, Fite, Philip Franklin, Fry, Gafafer, Gilman, W. C. Graustein, Gronwall, Grove, Haskins, Hausle, Hebbert, Robert Henderson, Hitchcock, Huntington, Dunham Jackson, Joffe, Kellogg, Kline, Lamson, Leavens, J. L. Love, McDonnell, McMackin, MacDuffee, J. B. Maclean, MacNeish, Mathews, H. H. Mitchell, Moir, Frank Morley, Mullins, Northcott, Oglesby, Osgood, Pfeiffer, Post, Press, Reddick, Reid, R. G. D. Richardson, Ritt, Ruger, Safford, Seely, P. F. Smith, Sosnow, Strong, Stuerm, Tanoue, J. S. Thompson, Wedderburn, Weisner, M. E. Wells, H. S. White, R. G. Wood, J. W. Young.

Thirty-five persons were elected to membership by the Council at this meeting as follows:

Dr. Anne Dale Biddle Andrews, University of California;
Professor Lloyd Collier Bagby, Ottawa University;
Professor Apolinario Baltazar, University of the Philippines;
Dr. John Douglas Barter, University of California;
Mr. John Biggerstaff, Reed College;
Professor Augustus Bogard, College of St. Teresa;
Mr. Garland Baird Briggs, University of Virginia;
Professor Frank E. Burcham, Central College;
Professor William Russell Burwell, University of Tennessee;
Mr. José Caminero, Havana;
Mr. Wendell Morris Coates, University of Michigan;
Professor Harold Athelstane Fales, Columbia University;
Professor Maurice Fréchet, University of Strasbourg;
Mr. Carl Arshag Garabedian, Harvard University;
Mrs. Eleanor Growe, University of California;
Professor Napoleon Bonaparte Heller, Temple University;
Professor Frank Kelso Hyatt, Pennsylvania Military College;
Professor Jesse Bythwood Jackson, University of South Carolina;
Professor Miles Abdel Keasey, Drexel Institute;
Professor Joseph Patrick Kelly, Boston College;
Mr. Arthur Louis Klein, California Institute of Technology;
Dr. Sophia Hazel Levy, University of California;

Mr. David A. Lunden, Cooper Union;
Professor Daniel Joseph Lynch, Boston College;
Professor James Thomas McCormick, Boston College;
Professor Patrick Joseph McHugh, Boston College;
Professor Pius Stephen Pretz, St. Benedict's College;
Miss Constance Rummons, University of Nebraska;
Professor Paul Gerhard Schmidt, St. Olaf College;
Professor William Henry Schuerman, Vanderbilt University;
Dr. Charles Donald Shane, University of California;
Professor George Waddel Snedecor, Iowa State College;
Miss Elizabeth Webb Wilson, Washington, D. C.;
Mr. Bing Chin Wong, University of California;
Professor Frank George Wren, Tufts College.

The Secretary also announced the election to membership, on January 16 (by special provision of the Council, under the suspension of By-Law I), of the following twenty-seven persons:

Mr. Rodney Whittemore Babcock, University of Wisconsin;
Mr. Raymond Walter Barnard, University of Michigan;
Professor Lawrence Henry Bowen, Furman University;
Mr. Martin Allen Brumbaugh, University of Pennsylvania;
Miss Elizabeth Carlson, University of Minnesota;
Professor Henry Clyde Carver, University of Michigan;
Miss Margaret Francis Chapman, Milwaukee State Normal School;
Professor Earl Church, Pennsylvania Military College;
Professor Charles Neblett Dickinson, Hollins College;
President Frances Augustine Driscoll, Villanova College;
Dean Robert Ryland Fleet, William Jewell College;
Mr. William Rogers Herod, General Electric Company, Schenectady;
Mr. Hemphill Hosford, Southern Methodist University;
Professor George Harold Jamison, Missouri State Teachers College, Kirksville;
Mr. Bradford Fisher Kimball, Harvard University;
Professor Paul Victor Levain, Holy Cross College;
Mr. James Lee Love, Gastonia Cotton Manufacturing Company;
Professor Ross W. Marriott, Swarthmore College;
Professor Ernest Alanson Pattengill, Iowa State College;
Professor William Richard Ransom, Tufts College;
Miss Margaret A. Shelley, College of St. Catherine;
Professor Hugo Ferdinand Sloctemyer, St. Louis University;
Mr. Sinclair Smith, California Institute of Technology;
Professor Arthur Dodd Snyder, Union College;
Professor Frederick Joseph Taylor, College of St. Thomas;
Mr. Wilmer Nichols Thompson, Drury College;
Professor Dudley Weldon Woodard, Howard University.

At the meeting of the Council, a letter was read from Professor E. V. Huntington reporting that an anonymous donor had offered to give the sum of $4,000 to print an additional volume of the TRANSACTIONS in 1922; the Council accepted the offer, with thanks to the donor and the intermediary. The Society also adopted a resolution of thanks for this very generous gift when it was announced by the Secretary at the morning session.

The Council appointed a committee consisting of Professors P. F. Smith (chairman), Eisenhart, and Hedrick, with power to arrange for the printing of the Society's publications; this committee was in particular authorized to proceed with the immediate publication of the additional volume of the TRANSACTIONS.

Professor C. N. Haskins was elected to succeed Professor L. E. Dickson as representative of the Society on the National Research Council for a period of three years beginning July 1, 1922.

On recommendation of the Council, the Society voted to extend the suspension of By-Law I until the end of April, and also to adopt the following substitute for Section 3 of By-Law II:

"Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with the principle that the life membership be regarded as the present worth of a life annuity due of a yearly payment equal to the net value of the annual dues, the annuity to be based on McClintock's 4 per cent *Tables for Annuitants* (*Male Lives*)."

Professor H. B. Fine presided at the morning session, relieved in the afternoon by Professors C. N. Haskins and H. S. White. The afternoon session was especially marked by a paper read, at the request of the programme committee, by Professor J. L. Coolidge, on *The basis of mathematical probability*. A number of members of the Actuarial Society were present, by invitation, to hear this paper.

Titles and abstracts of the papers read at this meeting follow below. Mr. Langer's paper was read by Professor Birkhoff, and Professor Schwatt's two papers by Professor

H. H. Mitchell. The papers of Professor Kellogg, Dr. Walsh, and Professors Emch, Morenus, and Hollcroft were read by title.

1. Professors G. D. Birkhoff and O. D. Kellogg: *Invariant points in function space.*

In this paper, the authors announce some of their results on the problem communicated to the Society on Dec. 30, 1920 (this BULLETIN, vol. 27 (1921), p. 307). The existence is established of invariant points in a region of n-space which is convex toward an interior point, under a continuous, one-valued transformation which carries points of the region into points of the region; also of the inverses of points on the hypersphere in n-space (n odd) with respect to a parametric transformation containing the identity. A number of analogous theorems are thus inferred for function space, first by a method of interpolation, and second, by a transition through a Hilbert space. Some theorems on linear transformations are derived directly. Applications are given to existence theorems for differential equations and integral equations, non-linear and linear. Incidentally, a simple proof is given of the important theorem that the general algebraic m-spread in n-space is non-singular, of which the authors have not found a demonstration in the literature.

2. Professor O. D. Kellogg: *A property of certain functions whose Sturmian developments do not terminate.*

In this note, the author reports the generalization to developments in terms of solutions of differential equations of the Sturm type of a result previously announced for Fourier series (see this BULLETIN, vol. 18 (1911), p. 234). A theorem on the rate of growth of the maximum of the absolute value of the nth derivative of an analytic function, on a circle interior to a region of analyticity, as a function of n, is also given.

3. Professor G. D. Birkhoff and Mr. R. E. Langer: *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order.*

This paper develops the theory of a system of n ordinary linear differential equations of the first order containing a parameter and subject to certain boundary conditions; the notation of matrices is used. The major portion of the treatment is devoted to the vector equation $\gamma'(x)\cdot = \{A(x)\lambda + B(x)\}\gamma(x)\cdot$

and to the system comprised of this equation and linear boundary conditions. The elements of $A(x)$ and $B(x)$ are assumed to be differentiable, and λ is a complex parameter. The asymptotic form of the solutions of the equation, and the distribution of the characteristic values of the system under regular and particular irregular conditions are obtained if the roots $\gamma_j(x)$ $(j = 1, 2, \cdots, n)$ of the equation $|a_{ij}(x) - \delta_{ij}\gamma(x)| = 0$ are non-vanishing and distinct, and satisfy the condition $\arg\{\gamma_j(x) - \gamma_i(x)\} = c_{ij}$ (c_{ij} a constant). Finally the formal development of a vector of arbitrary functions, $F(x)\cdot$, into a series of characteristic functions is derived and is proved to converge when the points $\gamma_j(x)$ $(j = 1, 2, \cdots, n)$ are either constrained to lie on a line through the origin of the complex plane (in particular are all real), or form the vertices of a polygon which, as x varies, at most expands or contracts about the origin.*

This paper will appear in the PROCEEDINGS OF THE AMERICAN ACADEMY.

4. Mr. R. E. Langer: *Developments associated with a boundary problem not linear in the parameter.*

It appears that the ordinary differential equations which have heretofore been used as sources of characteristic functions in expansion problems are each equivalent to a system of linear, first order equations of the type

$$g_i(x)u_i'(x) = L_i(u) + \lambda M_i(u) \quad (i = 1, 2, \cdots, n), \tag{1}$$

where $g_i \not\equiv 0$, while L_i and M_i are linear expressions in $u_1, \cdots, u_n$, $M_i \equiv 0$ for $i \neq n$, $M_n \not\equiv 0$.† The author discusses in this paper the expansion problem which results from choosing as a source of characteristic functions an equation which is equivalent to a system of type (1) in which $g_i \equiv 0$ for $i \neq n$, $g_n \not\equiv 0$, $M_i \not\equiv 0$ $(i = 1, 2, \cdots, n)$. The equation in question with boundary conditions is $P(\lambda)u'(x) = Q(\lambda, x)u(x)$, $u(a) = hu(b)$, $P(\lambda)$ and $Q(\lambda, x)$ being polynomials in λ which satisfy certain restrictions. It is shown that the characteristic values of the system cluster about certain points, and that n arbitrary functions may be formally developed with one determination of coefficients into series. These developments are shown to reduce under certain conditions to Fourier's expansions and are proved in every case (with usual restrictions on $f_i(x)$) to converge like Fourier's series.

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* TRANSACTIONS OF THIS SOCIETY, vol. 9 (1908), pp. 373–395.

† See a joint paper by Professor Birkhoff and the author, reported at this meeting, where a system in which $g_i \not\equiv 0$, $M_i \not\equiv 0$ $(i = 1, 2, \cdots, n)$, is treated.

5. Professor L. P. Eisenhart: *Ricci's principal directions for a Riemann space and the Einstein theory.*

In 1904 Ricci developed the idea of principal directions in a Riemann space of n dimensions. In doing so he introduced the contracted curvature tensor, which is fundamental in the Einstein theory, and thus gave a geometrical interpretation to it. The principal directions are those for which the mean curvature takes maximum and minimum values, the mean curvature at a point for a direction h being the sum of the Riemann curvatures for the $n - 1$ directions determined by h and each of $n - 1$ directions orthogonal to h. The present paper points out that a space in which the principal directions are completely indeterminate may be thought of as possessing a homogeneous character. Applying these considerations to spaces of four dimensions, it is shown that the three types of space chosen by Einstein in 1914, 1917 and 1919 as spaces free from matter are of this homogeneous character, and include all types of such spaces.

6. Dr. Jesse Douglas: *Normal congruences and quadruply infinite families of curves.*

The present paper deals with a general geometric problem suggested by a theorem of Thomson-Tait and a converse theorem of E. Kasner: to classify quadruply infinite families of curves in space, $y'' = F(x, y, z, y', z')$, $z'' = G(x, y, z, y', z')$, with respect to normal congruences contained within them. The analysis is based on the equations of variation of the differential equations of the family, and leads to an infinite system of partial differential equations of the Monge-Ampère type. The infinite system of equations is discussed as to its complete integrability, resulting in the following classification of families of ∞^4 curves: (1) ∞^∞ normal congruences, where the infinitude is that of all the surfaces of space. This is the case of the theorems of Thomson-Tait and Kasner, where the family is of the "natural" type. (2) ∞^∞ normal congruences, where the infinite exponent represents the generality of two arbitrary functions of one argument. (3) ∞^∞ normal congruences, where the infinite exponent corresponds to one arbitrary function of one argument. (4) ∞^3 normal congruences. (5) ∞^2 normal congruences. (6) No normal congruences, or a finite number, or an infinite number (which must be ∞^1 or ∞^∞), but all confined to a triply infinite sub-family. This is the general case.

7. Dr. T. H. Gronwall: *Qualitative properties of the ballistic trajectory. Second paper.*

In the last section of the author's first paper with this title (Annals of Mathematics (2), vol. 21 (1920), pp. 44–65), it is shown that under certain very broad assumptions on the G- and H-functions, the velocity of the projectile has only a finite number of maxima and minima. The present paper gives further information regarding the existence and location of these extremes.

8. Dr. T. H. Gronwall: *The reflection of X-rays in a finite number of equidistant parallel planes.*

In the Physical Review (May, 1921), Dr. Lamson has solved the corresponding, and simpler, problem for an infinite number of planes. The present paper reduces the problem for a finite number of planes to a non-linear difference equation of the second order, the solution of which is obtained algebraically.

9. Professor J. L. Coolidge: *The basis of mathematical probability.*

Certain historical and philosophic aspects of the mathematical foundations of the theory of probability were presented in this paper, which was read at the request of the programme committee.

10. Professor E. V. Huntington: *On the Alabama paradox in the problem of apportionment.*

If the problem of apportionment is regarded as a problem in minimizing some form of *total error*, there are 80 or more summation formulas to be considered. (See, for example, the writer's paper presented at the Wellesley meeting, September, 1921; this Bulletin, vol. 28 (1922), p. 15.) Two of these formulas are known to lead to the method of major fractions, and two to the method of equal proportions. It is here shown that all the other known summation formulas lead to the *Alabama paradox*, that is, an increase in the total size of the House may involve a decrease in the representation of one of the states.

(On the general problem of apportionment, see papers in the Quarterly Publication of the American Statistical Association, September 1921, and December, 1921.)

11. Professor E. V. Huntington: *On the d'Hondt method of apportionment, and its counterpart.*

If the problem of apportionment is regarded as a problem in minimizing the inequality between every state and every other state, there are five different comparison formulas to be considered, leading to five distinct methods of apportionment. The "amount by which the population of the over-populated state must be reduced to bring it down to parity with the other state," taken in the sense of absolute differences, is shown to lead to the d'Hondt method. The "amount by which the population of the under-populated state must be increased to bring it up to parity with the other state," taken in the sense of absolute differences, is shown to lead to the contra-d'Hondt method, a new method which favors the small states extremely, just as the d'Hondt method favors the large states extremely. But each of these tests, if taken in the sense of relative differences, leads to the method of equal proportions.

12. Dr. G. A. Pfeiffer: *Theorems on sequences of sets of points.*

The first theorem of this paper is the following: If the sum of the sequence of sets of points, $\{A_n\}$, is compact in a metric space, then there exists a subsequence of $\{A_n\}$ whose lower closed limit* set is identical with its upper closed limit set and is between the upper and lower closed limit sets of the given sequence. An obvious modification of the definitions and proof of this theorem gives an analogous theorem for sets of point sets. Essential use is made of the above theorem in proving the following theorems: (1) If the sum of a sequence of connected sets of diameter $\leqq \delta$ is compact in a metric space, then the lower closed limit set of the sequence is contained in a closed connected set (a continuum) which has a diameter $\leqq \delta$ and which is contained in the upper closed limit set of the sequence. (2) If the sum of a sequence of continua $\{A_n\}$ is compact in a metric space and if A_n and A_{n+1} have a point in common and $\Sigma\delta_n$, where δ_n is the diameter of A_n, is convergent, then the closed limit set of $\{A_n\}$ contains one and only one point.

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* Lower (upper) closed limit set ≡ untere (obere) abgeschlossene Limes. See Hausdorff, *Grundzüge der Mengenlehre*, p. 236.

13. Professor C. A. Fischer: *The Fredholm theory of Stieltjes integral equations.*

A large part of the Fredholm theory has been developed for the equation $f(x) = \varphi(x) - \int_a^b \varphi(y) d_y K(x, y)$, by F. Riesz, where the integral defines a completely continuous transformation, without making use of the Fredholm determinant. In a previous paper Professor Fischer has found necessary and sufficient conditions which $K(x, y)$ must satisfy, in order that the transformation be completely continuous. In the present paper, modified definitions of Stieltjes integrals are given, and a set of conditions for $K(x, y)$ is found, under which the Fredholm determinant is defined for the above equation, and the classical Fredholm theory is applied directly. This theory neither includes, nor is included by, the Riesz theory.

14. Dr. J. L. Walsh: *A closed set of normal orthogonal functions.*

This paper presents a simple example of a closed normal orthogonal set of functions on the interval (0, 1). Each function takes on no value other than $+ 1$ or $- 1$, except at a finite number of points; the nth function has precisely $n + 1$ zeros (that is, sign-changes) in the interior of the interval; and a large class of arbitrary functions can be expanded in terms of these functions. The set is somewhat analogous to one considered by Haar, but has properties more closely allied to those of the more familiar sets of orthogonal functions.

15. Professor Arnold Emch: *Kinematics in a complex plane and some geometric applications.*

This paper appears in this number of this BULLETIN.

16. Professor J. F. Ritt: *On functions with integrals of elementary character.*

Let w be an integral of any algebraic function of z. It is a result of Liouville's that if w can be expressed in terms of elementary functions, then

$$(1) \qquad w = \alpha_0 + c_1 \log \alpha_1 + c_2 \log \alpha_2 + \cdots + c_n \log \alpha_n,$$

where each α is an algebraic function and each c a constant. The present paper gives an extension of Liouville's result. It is shown that if w is a solution of an elementary transcendental equation, then w is of the form (1). It follows that no elliptic function is a solution of an elementary transcendental equation. The same fact can be proved with respect to the integral logarithm and to many other transcendental functions.

17. Professor Eugenie M. Morenus: *Geometric properties of the system of all the curves of constant pressure in a plane field of force.*

The problem here considered is: In a plane field of force, to find smooth curves along which a constrained motion is possible such that the pressure of a moving particle against the curve shall remain constant. For one required constant pressure c there are ∞^3 such curves in the plane. The triply infinite system S_c has four geometrical properties which are shown to be sufficient to identify a set of ∞^3 curves as curves of constant pressure in a field of force. The quadruply infinite set of curves obtained by employing all possible values of c has eight geometrical properties which are sufficient to characterize completely a set of ∞^4 curves as curves of constant pressure. Eight additional properties of the quadruply infinite set are demonstrated.

18. Professor W. C. Graustein: *Spherical representation of conjugate systems and asymptotic lines.*

It is well known that a system of curves on the Gauss sphere represents a conjugate system of curves on each of infinitely many surfaces. It is here shown that, if the ratio of the radii of normal curvature in the conjugate directions is prescribed subject to a certain condition, one of the required surfaces is determined to within its homothetics. The general result is peculiarly adaptable to the important special cases of conjugate systems which have equal point or equal plane invariants or are isothermal-conjugate. It also lends itself readily to the treatment of lines of curvature. By similar methods applied to asymptotic lines a generalization of the theorem of Dini, giving the condition under which a system of curves on the sphere is the spherical representation of the asymptotic lines on a surface, is obtained. The paper will appear in the ANNALS OF MATHEMATICS.

19. Mr. Charles Manneback: *The distribution of current in a long cylindrical conductor.*

The problem of determining the distribution of density of an alternating current in a cylindrical conductor under the influence of a remote return parallel wire leads to the well known partial differential equations of Laplace and Poisson, with boundary conditions. The solution of the problem has not received hitherto a satisfactory answer. In the present paper the equivalent integral equation

$$i(r,\theta) = \frac{\alpha^2 I}{2\pi}\log\frac{\delta}{D} + \frac{\alpha^2}{2\pi}\iint_s i\,(\rho,\phi)\log\frac{d}{\rho}\,dS\,(\rho,\phi)$$

is set up and the following solution, which is a uniformly convergent series,*

$$i(r,\theta) = -\sum_{n=1}^{\infty}\frac{I}{\pi a^2}\left(\frac{a}{\delta}\right)^n j\alpha a\frac{J_n(j\alpha r)}{J_{n-1}(j\alpha a)}\cos n\theta,$$

is derived. Several related problems are also treated. The method appears capable of quite wide application.

20. Mr. A. Press: *Operational solution of equations of nth degree.*

An equation of the form $f(y) = a$ can be solved operationally if a mathematical operator f^{-1} can be found such that operating on both sides of the equation gives $f^{-1}\cdot f(y) = f^{-1}(a) = y$; f^{-1} is the inverse functional form of f. The operator multiplication (X) of ordinary algebra provides the basis of the required functional form with the understanding that $X^m\cdot y = f(y) = y^m$. The f in an equation of the nth degree then takes the form $f = a_mX^m + a_{m-1}X^{m-1} + \cdots = \Sigma a_mX^m$. The corresponding inverse of f is assumed of the form $f^{-1} = \Sigma a_pX^p$ which may be an ascending or descending series of implied multiplications or roots. By developing the corresponding algebra of the multiplication X as an operator either the inversion of series is accomplished or the form of resolution first obtained by McClintock. The operational developments in ascending form give the real roots and in descending form the complex roots. The algebra is particularly valuable in determining the complex roots of Heaviside's determinantal equations.

21. Professor T. R. Hollcroft: *Maximal cuspidal curves.*

A maximal cuspidal curve is an algebraic plane curve which has the greatest number of cusps possible for a given order n and genus $p \leqq \frac{1}{6}(n-2)(n-3)$. For $p > \frac{1}{6}(n-2)(n-3)$ all the double points may be cusps so there is no limit to the number of cusps except the genus. First, formulas are found giving the singularities of maximal cuspidal curves from genus 0 to p_0, the genus of the curve of lowest genus of the minimum class m_0, and the number of curves of class m_0 (which are always maximal cuspidal curves) and their singularities. Next, beginning with the self-dual maximal cuspidal curves, formulas are derived giving the number of maximal cuspidal curves for each class from n down to m_0. From these the maximum and minimum genus of each class is found, thence an inequality giving the class of any maximal cuspidal curve of given order and genus $p > p_0$, and from this an inequality giving the maximum number of cusps for any algebraic curve of given order and genus $p > p_0$. Using this inequality together with the one found for $p \leqq p_0$, we have the following: The maximum number of cusps for any algebraic plane curve of order n and genus $p \leqq \frac{1}{6}(n-2)(n-3)$ is given by the greatest integer k satisfying the smaller of these two inequalities when both are real or the first when the second is imaginary: (1) $k \leqq \frac{3}{2}(n-2)+3p$; (2) $k \leqq 2(n+p) - \frac{1}{2}(11+\sqrt{24p-8n+25})$.

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* r, θ and ρ, ϕ = polar coordinates of points of the conductor's cross-section; I = total return current; i = density of induced current in the conductor; a = radius of the conductor; δ = distance between centers of the parallel wires; D = distance between the return wire and point r, θ; d = distance between points r, θ and ρ, ϕ; dS = element of the conductor's cross-section area; α = a complex quantity of argument $\pi/4$.

22. Professor I. J. Schwatt: *Method for the separation into partial fractions of powers of trigonometric functions.*

The separation into partial fractions of $\tan^p x$, $\operatorname{ctn}^p x$, $\sec^p x$ and $\csc^p x$ depends on the evaluation of expressions like

$$\left(1 \pm \frac{2x}{(2n+1)\pi}\right)^p \tan^p x, \quad x = \mp \frac{2n+1}{2}\pi;$$

$$\left(1 \pm \frac{x}{n\pi}\right)^p \operatorname{ctn}^p x, \quad x = \mp n\pi; \text{ etc.}$$

The methods devised and the results obtained in this paper are believed to be new.

23. Professor I. J. Schwatt: *The expansion of the continued product* $\Pi\,(x+k)$.

The methods developed in the preceding paper have enabled the author to obtain the general term of the expansion of the continued product $\Pi\,(x+k)$.

R. G. RICHARDSON,
Secretary

A PROPERTY OF CONTINUITY*

BY D. C. GILLESPIE

If ξ and η are two points of the interval (a, b) in which the function $f(x)$ is continuous, then the function takes on all values between $f(\xi)$ and $f(\eta)$ as x changes from ξ to η. This property, which we shall designate by (A), is common to all continuous functions, but it is possessed also by other functions. It was shown, for example, by Darboux† that all derived functions possess property (A), and it was pointed out by Lebesgue‡ that still other functions which are neither continuous nor derived have the property.

The present note is concerned with functions having property (A). The character of the discontinuities that such a function may have is shown. Additional conditions which are sufficient to insure that the functions be continuous or continuous and monotone follow. A theorem stating that when all the discontinuities of a function are of a certain kind it has property (A) is proved, and a function having property (A) and having its set of points of continuity and its set of points of discontinuity each everywhere-dense is constructed.

If c is a point of discontinuity of a function $f(x)$ having property (A), then in any interval about c the function takes on all values between its maximum§ and minimum at c. If $f(x)$ is unbounded, for example from above, in the neighborhood of c, the function could have no maximum at c, in which case, obviously, $f(x)$ would have, in any interval about c, all values greater than its minimum at c.

As an immediate consequence of the character of the discontinuities of a function having property (A), it follows that *such a function will be continuous unless the set of values it assumes an infinite number of times fills at least one interval.* The converse of this statement is not true. An examination of the Weierstrass non-differentiable continuous function reveals the fact that it takes on each value between its value at 0 and its value at 1 an infinite number of times for values of the variable in the interval $(0, 1)$.

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* Presented to the Society, Sept. 7, 1920.

† ANNALES DE L'ECOLE NORMALE (2), vol. 4 (1875), pp. 109, 110.

‡ *Leçons sur l'Intégration*, p. 92.

§ Hobson, *Theory of Functions of a Real Variable*, p. 234, for definition of maximum and minimum of a function at a point.

If then a function having property (A) *takes on no value more than once, it is continuous and hence also monotone.* This idea is capable of a slight extension and then comes to an abrupt end. Since a function having property (A) and taking on no value an infinite number of times is continuous, we shall state the extension for continuous functions.

If $f(x)$, *continuous in* (a, b), *assumes no value more than twice, it is possible to divide the interval into three parts in each of which* $f(x)$ *is monotone.*

It will be convenient to consider two cases:

(1) $f(a) = f(b)$,

(2) $f(a) \neq f(b)$.

In case (1), $f(x)$ must have either its greatest or its least value at a point ξ within (a, b). In each of the intervals (a, ξ) and (ξ, b) the function takes on every value between $f(a) = f(b)$ and $f(\xi)$ one and therefore only one time, hence in each of these intervals $f(x)$ is monotone.

In case (2) assume $f(a) < f(b)$. Suppose now $f(x)$ has its least value at a point ξ and its greatest value at a point η, where ξ and η are, of course, not necessarily distinct from a and b.* It follows from the assumption $f(a) < f(b)$ that $\xi < \eta$, for if η were less than ξ, the function would have the value $f(b)$, for example, once in the interval (a, η), once in the interval (η, ξ) and at b. In the interval (a, ξ) the function has its least value at ξ and it must have its greatest value at a, since, if the function were greater than or equal to $f(a)$ within (a, ξ), it would assume the value $f(a)$ at least three times: once in the interval (ξ, η), once within (a, ξ) and at a. This function having in the interval (a, ξ) its greatest value at a and its least value at ξ, can take on no value two times in (a, ξ). For suppose the function has the value M at two points p_1 and p_2 within (a, ξ), then, unless $f(x) = M$ has a root within (p_1, p_2), either $f(x) > M$ for all points within (p_1, p_2) and $f(x)$ has all values between M and its greatest value in (p_1, p_2) at least twice in the interval (p_1, p_2) and once in the interval (a, p_1); or $f(x) < M$ for all points within (p_1, p_2) and $f(x)$ has all values between M and its smallest value in (p_1, p_2) at least twice in (p_1, p_2) and once in (p_2, ξ). The function then has each value between $f(a)$ and $f(\xi)$ one and only one time in (a, ξ) and is therefore monotone. In the same way one shows that $f(x)$ is monotone in (ξ, η) and (η, b), the extreme values of the function for the intervals occurring at the end-points of the intervals.

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* If $f(x)$ have an extreme value at a or b, take $\xi = a$ or $\eta = b$.

If we assume that $f(x)$ takes on no value more than three times and that $f(a) = f(b)$, a similar argument will show that the interval may be divided into three parts in each of which the function is monotone.

This idea is capable of no further extension, since a continuous function may take on no value more than three times in an interval and yet be such that it is impossible to divide the interval into a finite number of parts in which the function is monotone. This statement is verified by the example:

$f(x) = \pi x + x^2 \sin(\pi/x)$ for $0 < x \leqq 1$,

$f(0) = 0$.

The function has proper maximum points at $1/2, 1/4, 1/6, \cdots$ and proper minimum points at the roots of the equation $\tan(\pi/2x) = -(2x/\pi)$, which occur in the intervals $(1/2, 1)$, $(1/4, 1/3)$, $(1/6, 1/5)$, etc. In the interval $[1/(2n+1), 1/(2n-1)]$ the function has one maximum point and one minimum point, hence takes on no value more than three times in this interval. Moreover,

$$\frac{\pi}{2n+1} \leqq f(x) \leqq \frac{\pi}{2n-1}$$

for

$$\frac{1}{2n+1} \leqq x \leqq \frac{1}{2n-1},$$

$$f(x) > \frac{\pi}{2n-1} \quad \text{for} \quad x > \frac{1}{2n-1}$$

and

$$f(x) < \frac{\pi}{2n+1} \qquad \text{for} \qquad x < \frac{1}{2n+1}\,.$$

We have shown that a function having property (A) assumes in any interval about a point of discontinuity every value between its maximum and minimum at this point. The converse of this is not true; for example, $f(x) = 1 + x + \sin(\pi/x)$ for $0 < x \leqq 1$, $f(0) = 1$ and $f(x) = x$ for $-1 \leqq x < 0$ has at zero its only discontinuity. In every interval about zero it assumes every value between 0, its minimum at 0, and 2, its maximum at 0, and although $f(-1) = -1$ and $f(1) = 2$, $f(x) = 0$ has no root in $(-1, 1)$. It is to be observed that the values between 0 and 2 are not assumed by the function *in any interval whose upper end-point is* 0.

THEOREM. *If a function $f(x)$ at every point of discontinuity c takes on all values between its maximum and minimum at c in every interval of which c is an end-point, then the function has property* (A).

Suppose ξ_1 and η_1, $\xi_1 < \eta_1$, are two points of the interval (a, b) in which $f(x)$ is defined, and that $f(\xi_1) < M < f(\eta_1)$; we shall show that $f(x) = M$ has a root between ξ_1 and η_1. Consider the set of points in (ξ_1, η_1) at which $f(x) > M$ and take the lower limit p_1 of this set. If $p_1 = \xi_1$, then, since ξ_1 is the lower limit of a set of points at which $f(x) > M$, the maximum of $f(x)$ at ξ_1 is not less than M. If this maximum of $f(x)$ at ξ_1 is greater than M, then $f(x) = M$ has a root in every interval of which ξ_1 is an end-point. If $p_1 > \xi_1$, then there is no point within (ξ_1, p_1) for which $f(x) > M$, hence the maximum of $f(x)$ at p_1 is M. In any case, whether p_1 is equal to ξ_1 or greater than ξ_1, the equation $f(x) = M$ either has a root in (ξ_1, η_1) or $f(p_1) < M$ and the maximum of $f(x)$ at p_1 is M. Suppose $M - f(p_1) = \epsilon$; now choose a positive $\delta < [(\eta_1 - p_1)/2]$ and such that $f(x) < M + \epsilon/2$ in the interval $(p_1, p_1 + \delta)$. Within this interval there is a point η_2 at which $f(\eta_2) > M$. Now it may be established likewise that either $f(x) = M$ has a root in (p_1, η_2) or there exists at the upper end of (p_1, η_2) a subinterval (ξ_2, η_2) such that $f(\xi_2) < M$ and $f(x) > M - (\epsilon/2)$. Thus in the interval (ξ_2, η_2)

$$|f(x) - M| < \epsilon/2.$$

Continuing this process, we obtain a sequence of intervals (ξ_n, η_n) each lying within the preceding one and of length less than $(\eta_1 - \xi_1)/2^{n-1}$ for which $f(\xi_n) < M$, $f(\eta_n) > M$, $|f(x) - M| < \epsilon/2^{n-1}$ for x in (ξ_n, η_n). Hence at the one point p common to this set of intervals $f(p) = M$.

It will require only obvious changes in the preceding argument to prove a more general theorem*. Using the four symbols $\overline{f}(c + 0)$, $\underline{f}(c + 0)$, $\overline{f}(c - 0)$ and $\underline{f}(c - 0)$ to designate the upper and lower limits† of $f(x)$ to the right and left of c respectively, this theorem may be stated:

If at every point of discontinuity c of $f(x)$, $\underline{f}(c + o) \leqq f(c) \leqq \overline{f}(c + 0)$, $\underline{f}(c - 0) \leqq f(c) \leqq \overline{f}(c - 0)$ and $f(x)$ takes on every value between $\underline{f}(c + 0)$ and $\overline{f}(c + 0)$ in every interval whose left end-point is c and every value between $\underline{f}(c - 0)$ and $\overline{f}(c - 0)$ in every interval whose right end-point is c, then $f(x)$ has property (A).

The conditions thus imposed upon $f(x)$ are not only sufficient but also clearly necessary for the existence of property (A).

As illustrations of functions having property (A) we have on the one hand continuous function or certain functions with a finite number of discontinuities, and on the other hand totally discontinuous functions.‡

The following function § defined by a uniformly convergent series has property (A) and has its set of points of continuity and its set of points of discontinuity each everywhere-dense: ||

$$f(x) \| = \sum_{n=1}^{\infty} \frac{1}{n^2} g(x - r_n),$$

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* E. R. Hedrick pointed out to me that the argument really proved this more general theorem.

† Hobson, loc. cit., p. 231.

‡ Lebesgue, loc. cit., p. 90.

§ Pierpont, *Theory of Functions of a Real Variable*, Vol. II, p. 463, Ex. 1.

|| W. A. Hurwitz suggested the probability of the function defined possessing the properties enumerated. The present note grew out of a study of this example.

where $r_1, r_2, r_3, \cdots$, denotes the set of rational numbers, $g(x) = \sin(\pi/x)$ for $x = 0$ and $g(0) = 0$.

The function $f(x)$ is continuous at all irrational points and has a discontinuity of the second kind with oscillation equal to $2/n^2$ at each rational point r_n. To prove the function has property (A), it is sufficient, in consequence of the theorem just established, to show that $f(x) = M$, where $f(r_n) - (1/n^2) < M < f(r_n) + 1/n^2$, has a root in every interval of which r_n is an end-point. From the definition of $f(x)$ it is obvious that corresponding to any positive number ϵ there exists a positive number δ such that the oscillation of $f(x)$ at any point lying inside the interval $(r_n, r_n + \delta)$ is less than ϵ. Within $(r_n, r_n + \delta)$ there is a point η_1 at which $f(\eta_1) > M$ and then within (r_n, η_1) a point ξ_1 at which $f(\xi_1) < M$. By taking the upper limit of those points in (ξ_1, η_1) at which $f(x) < M$ and proceeding as before, a convergent sequence of intervals (ξ_m, η_m) where $f(\xi_m) < M < f(\eta_m)$ and for which the oscillation at each point is less than $\epsilon/2^m$ may be constructed. At the one point p common to the set of intervals, $f(p) = M$. The same is true, of course, of intervals bounded above by r_n.

One could infer, from facts already established, the existence of a function having property (A) and having its set of points of continuity and its set of points of discontinuity each everywhere-dense. For functions which are everywhere oscillating and which nevertheless have a derivative at every point of an interval have been constructed.* The derived function has, of course, property (A); this derived function being the limit of a sequence of continuous functions has its points of continuity everywhere-dense;† its points of discontinuity are also everywhere-dense since it is both positive and negative in every interval.

CORNELL UNIVERSITY

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* Köpcke, MATHEMATISCHE ANNALEN, vols. 34 and 35.

† Baire, *Leçons sur les Fonciions Discontinues*, p. 98.

KINEMATICS IN A COMPLEX PLANE AND SOME GEOMETRIC APPLICATIONS *

BY ARNOLD EMCH

1. *Introduction.* From an elementary standpoint one is apt to consider geometry in a complex plane as an accessory of heuristic value for function theory. In a more advanced sense one recognizes the fundamental importance and intrinsic value of the geometric problem of partition of the complex plane by circular arcs in connection with the properties and classification of certain linear substitution groups, with the corresponding automorphic functions, and, in particular, with the theory of algebraic curves and their Riemann surfaces and uniformization. But even in a more elementary sense, the complex plane is the natural medium for the solution of certain specific geometric problems. As an example may be mentioned the "geometry of the polynomial," involving the theory of stelloid and circular curves and their focal properties.†

Also a number of problems in geometric kinematics may be solved conveniently in a complex plane, as has been shown by Koenigs,‡ Study,§ and others. In the present paper I shall show by further examples of this kind the simplicity and elegance of the complex treatment.

2. *Similar Triangles.* As can easily be verified, a necessary and sufficient condition for the equi-sensed similitude of two triangles $z_1z_2z_3$, $z_1'z_2'z_3'$ is the vanishing of the determinant

$$(1) \qquad \begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0.$$

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* Presented to the Society, Feb. 25, 1922.

† See Emch, *On a certain generation of rational circular and isotropic curves*, this BULLETIN, vol. 25, pp. 397–404 (1919), and also *On plane algebraic curves with a given system of foci*, same volume, pp. 157–161.

‡ *Leçons de Cinématique: Les imaginaires dans la cinématique du plan*, pp. 324–332 (1897).

§ *Vorlesungen über ausgewählte Gegenstände der Geometrie.* Erstes Heft: *Ebene analytische Kurven und zu ihnen gehörige Abbildungen*, pp. 1–18 (1911).

Denoting by $\bar{z}$ the conjugate of z, two triangles $z_1z_2z_3$, $z_1'z_2'z_3'$ are similar with the sense inverted when

$$\begin{vmatrix} \bar{z}_1 & z_1' & 1 \\ \bar{z}_2 & z_2' & 1 \\ \bar{z}_3 & z_3' & 1 \end{vmatrix} \equiv 0. \tag{2}$$

For an equilateral triangle $z_1z_2z_3$ we simply have to set up the condition that the triangle $z_2z_3z_1$ in the same order be similar to the triangle $z_1z_2z_3$; i.e.,

$$\begin{vmatrix} z_1 & z_2 & 1 \\ z_2 & z_3 & 1 \\ z_3 & z_1 & 1 \end{vmatrix} \equiv 0.$$

This reduces to

$$z_1^2 + z_2^2 + z_3^2 - z_2z_3 - z_3z_1 - z_1z_2 = 0 \tag{3}$$

as a necessary and sufficient condition that the triangle $z_1z_2z_3$ be equilateral.

3. *The Group of Movements in a Plane.* The linear transformation

$$z' = az + b \tag{4}$$

may be considered as a movement (including uniform dilatation) of the z-plane into a new position indicated by z' and referred to the same system of coordinates (original z-plane). Putting $a = r_1e^{i\alpha_1}$, $b = r_2e^{i\alpha_2}$, $z = re^{i\theta}$, (4) becomes

$$z' = r_1re^{i(\theta+\alpha_1)} + r_2e^{i\alpha_2}, \tag{5}$$

and it is seen at once that the movement is equivalent to the effect of the succession of substitutions

(a) S_1(*similitude*): $z_1 = r_1 \cdot z$;
(b) S_2(*rotation*): $z_2 = z_1 \cdot e^{i\alpha_1}$;
(c) S_3(*translation*): $z_3 = z' = z_2 + r_2e^{i\alpha_2}$.

The totality of all movements (4) forms a continuous projective four-parameter group and contains (a), (b), (c) as subgroups. The invariant points in the movement from the z- to the z'-plane are $z = b/(1 - a)$ and $z = \infty$.

When z describes a figure in the original plane, z' describes a similar figure in the displaced plane with the coefficient of dilatation equal to r_1. In particular, when z describes either a straight line or a circle, z' will describe a straight line or a circle. Moreover, since

$$dz' = a \cdot dz, \qquad \text{and} \qquad |dz'| = r_1 \cdot |dz|,$$

the ratio of the velocities of z and z' is constant. Thus when z describes a straight line with uniform velocity, z' will describe a straight line with uniform velocity. When z describes a circle, z' describes a circle with the same angular velocity, i.e., z and z' describe their respective circles in the same time.

4. *On the Movement of a Triangle in Deformation and Remaining Similar to Itself.* When z describes the unit-circle $z = e^{i\theta}$,

$$z' = r_1 e^{i(\theta+\alpha_1)} + r_2 e^{i\alpha_2} \tag{6}$$

describes obviously a circle with $r_2 e^{i\alpha_2}$ as a center and r_1 as a radius. Consider now the triangle formed by the origin, the unit-point on the real axis and the fixed point $\rho e^{i\beta}$. On the line zz' which connects the point z on the unit-circle to the point z', corresponding to z by (4), erect a triangle $z_3 z z'$ similar, in the same order, to the triangle $\rho e^{i\beta}$, 0, 1. Then

$$\begin{vmatrix} z_3 & \rho e^{i\beta} & 1 \\ e^{i\theta} & 0 & 1 \\ r_1 e^{i(\theta+\alpha_1)} + r_2 e^{i\alpha_2} & 1 & 1 \end{vmatrix} = 0.$$

From this

$$-z_3 + e^{i\theta} - \rho e^{i(\theta+\beta)} + \rho r_1 e^{i(\theta+\alpha_1+\beta)} + \rho r_2 e^{i(\alpha_2+\beta)} = 0;$$

$$z_3 = \{\rho r_1 e^{i(\alpha_1+\beta)} - \rho e^{i\beta} + 1\} e^{i\theta} + \rho r_2 e^{i(\alpha_2+\beta)}. \tag{7}$$

As (7) has the same form as (6) and θ is the only variable, z_3 clearly describes a circle with $|\rho r_1 e^{i(\alpha_1+\beta)} - \rho e^{i\beta} + 1|$ as a radius and $\rho r_2 e^{i(\alpha_2+\beta)}$ as a center. Obviously, z_3, z', z describe the circles with the same angular velocities, i.e., they simultaneously describe the circles completely. Denote the centers of the circles described by z, z', z_3 by c_1, c_2, c_3, so that $c_1 = 0$, $c_2 = r_2 e^{i\alpha_2}$, $c_3 = \rho r_2 e^{i(\alpha_2+\beta)}$. Comparing this triangle with the original fixed triangle 0, 1, $\rho e^{i\beta}$, we see that

$$\begin{vmatrix} 0 & 0 & 1 \\ r_2 e^{i\alpha_2} & 1 & 1 \\ \rho r_2 e^{i(\alpha_2+\beta)} & \rho e^{i\beta} & 1 \end{vmatrix} \equiv 0.$$

From this it follows that the two triangles are similar.

Consider next a point z_4 which remains always similarly attached to the triangle $zz'z_3$, so that also $zz'z_4$ remains similar to itself. But such a triangle $zz'z_4$ arises from a definite fixed similar triangle 0, 1, $\delta e^{i\nu}$, just as $zz'z_3$ arises from 0, 1, $\rho e^{i\beta}$. Hence also z_4 describes a circle, and $\triangle c_1c_2c_4 \sim \triangle zz'z_4$, i.e., if a fourth point z_4 remains similarly attached to the triangle zz_1z_3, then z_4 describes a circle whose center c_4 is similarly attached to $c_1c_2c_3$. The same situation prevails for other similarly attached points $z_5, z_6, \cdots$, and as the geometric result does not depend upon the orientation of the complex plane, we may state the following theorem.

THEOREM 1. *If a variable closed or unclosed polygon*

$$(z) = z_0z_1z_2 \cdots z_n$$

of $(n + 1)$ *vertices remains similar to some arbitrary fixed polygon*

$$(a) = a_0a_1a_2 \cdots a_n,$$

and any two of its vertices describe two fixed circles with the same angular velocities, then all other vertices describe circles with the same angular velocities. The polygon

$$(c) = c_0c_1c_2c_3 \cdots c_n$$

of the centers of these circles is similar to the fixed polygon (a).

From this follows obviously the corollary:

COROLLARY.* *If any two vertices of a regular variable polygon* $(z) = z_1z_2 \cdots z_n$ *of* n *vertices and center* z_0 *describe two circles with the same angular velocities, then the remaining* $n - 2$ *vertices describe circles with the same angular velocities. The centers* $c_1c_2c_3 \cdots c_n$ *of these circles form a fixed regular polygon* (c), *whose center* c_0 *is the center of the circle described by the center* z_0 *of the variable polygon* (z).

5. *Case of Movements on Straight Lines.* It is not difficult to show that the theorem of the preceding section holds, when two points of the polygon (z) describe straight lines with uniform velocities. We may restate the result.

THEOREM 2. *If any two points of a variable polygon* (z), *which remains similar to some fixed polygon* (a), *describe two straight lines with uniform velocities, then all other points of* (z) *describe straight lines with uniform velocities.*

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* This contains Study's theorem on the square, loc. cit., p. 16, as a particular case. It is obviously not necessary to make a restriction in the choice of the two vertices as Study does.

In particular, when (z) *is a regular polygon, its center* z_0 *also describes a straight line.*

We propose next to discover whether it is possible to find triangles with their vertices on three arbitrarily given fixed lines and similar to a fixed triangle $0, 1, re^{ia}$. Let $\lambda_1, \lambda_2, \lambda_3$ be three variable parameters; then the three lines may be written in the form

$$\begin{aligned} z_1 &= a_1 + \lambda_1 e^{i a_1}, \\ z_2 &= a_2 + \lambda_2 e^{i a_2}, \\ z_3 &= a_3 + \lambda_3 e^{i a_3}. \end{aligned}$$

For every set of values $\lambda_1, \lambda_2, \lambda_3$ the three points z_1, z_2, z_3 form a triangle on the three lines. Similitude to the fixed triangle requires

$$\begin{vmatrix} a_1 + \lambda_1 e^{i a_1} & 0 & 1 \\ a_2 + \lambda_2 e^{i a_2} & 1 & 1 \\ a_3 + \lambda_3 e^{i a_3} & re^{ia} & 1 \end{vmatrix} = 0.$$

Expanding this and separating real from imaginary, we get an expression of the type

$$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 + A_4 + i(B_1\lambda_1 + B_2\lambda_2 + B_3\lambda_3 + B_4) = 0$$

in which the A's and B's and λ's are real. In order that this be satisfied for real values of the λ's, there must be

$$\begin{aligned} A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 + A_4 &= 0, \\ B_1\lambda_1 + B_2\lambda_2 + B_3\lambda_3 + B_4 &= 0, \end{aligned}$$

which shows the *existence of a simply infinite set* of solutions $(\lambda_1, \lambda_2, \lambda_3)$, or of such triangles. Hence we have the following theorem.

THEOREM 3. *If three lines* l_1, l_2, l_3 *are given, it is always possible to describe them simultaneously by three points of a one-parameter variable triangle which remains similar to a fixed triangle. The vertices of the triangle describe the sides* $l_1 l_2 l_3$ *with definite uniform velocities.*

Now choose a fourth point z_4 so that the variable quadrangle $z_1 z_2 z_3 z_4$, with z_1, z_2, z_3 moving on $l_1 l_2 l_3$, remains similar to a fixed quadrangle $a_1 a_2 a_3 a_4$; then z_4 describes a straight line l_4.

If four lines $l_0l_1l_2l_3$ are given, and we let z_1, z_2, z_3 describe l_1, l_2, l_3 as before, then z_4 describes a line l_4. When z_4, moving on l_4, reaches the intersection z_0' of l_4 with l_0, z_1, z_2, z_3 will have reached certain positions $z_1'z_2'z_3'$ on l_1, l_2, l_3, respectively, so that $z_0'z_1'z_2'z_3' \sim a_1a_2a_3a_4$.

THEOREM 4. *It is always possible to inscribe in a given quadrilateral a quadrangle similar to a given quadrangle. In particular, it is always possible to inscribe a square in a given quadrangle.**

Theorem I is also true when one of the two circles determining the movement degenerates into a point circle, or when one degenerates into a point circle and the other into a straight line. Thus follows

THEOREM 5. *If a variable triangle $z_1z_2z_3$ remains similar to a fixed triangle, and z_3 is stationary, while z_1 describes a straight line with uniform velocity, then z_2 describes another straight line with uniform velocity.*

As a particular case assume an isosceles right triangle $z_1z_2z_3$ with $z_3 = ei$ fixed on the axis of imaginaries and $z_1 = \lambda$, the vertex of the right angle, describing the axis of reals. Then $z_2 - \lambda = -(ei - \lambda)i$, and $z_2 = e + \lambda(1 + i)$, i.e., z_2 describes a line through the point e on the real axis and making an angle of 45° with the positive part of the real axis. This result leads to the following method.

6. *Inscribing a Square in a Quadrilateral.* Let $l_1l_2l_3l_4$ be the sides of a proper quadrilateral following each other in the positive sense. On l_1 choose a trial point A_1 and drop a perpendicular A_1Q_1 on l_2. From Q_1 measure off on l_2 in the positive sense $Q_1A_2' = A_1Q_1$. Through A_2' draw a line g_1 making an angle of 45° with the positive direction of l_2, and intersecting l_3 in A_3. Then A_1A_3 form two opposite vertices of a square on $l_1l_2l_3$ and we can easily find the other vertices A_2, A_4. Repeat the same construction, starting with a second

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* The second part of this theorem is well known. See Dr. Hebbert's papers, ANNALS OF MATHEMATICS (2), vol. 16, pp. 38–42; 61–71 (1914–15), and references given there.

trial point B_1 on l_1, resulting in a second square $B_1B_2B_3B_4$ with $B_1B_2B_3$ on $l_1l_2l_3$. Now when a variable square moves with three vertices on three lines $l_1l_2l_3$, the fourth vertex describes a line l_4'. $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are two positions of this square, and A_4 and B_4 determine the line l_4'. Hence, the point of intersection P_4 of l_4' and l_4 is a vertex of the inscribed square. Repeating the above construction in the reversed order, starting with P_4 on l_4, easily gives the square $P_4P_3P_2P_1$ inscribed in $l_1l_2l_3l_4$.

This however is not the only square. A second square is obtained by measuring off $Q_1A_2^* = - Q_1A_2$ in the negative sense, by drawing g_1^* through A_2^*, making an angle of 45° with the negative direction of l_2, and proceeding with the construction as explained above. Starting again with l_1 and choosing the order $l_1l_2l_4l_3$, there are again two squares whose vertices lie in succession on $l_1l_2l_4l_3$. Likewise there are two squares for the order $l_1l_3l_2l_4$. The remaining three orders $l_1l_4l_3l_2$, $l_1l_3l_4l_2$, $l_1l_4l_2l_3$ are cyclic substitutions of the above three and do not produce any new squares.

THEOREM 6. *There are, in general, six squares that may be inscribed in a proper quadrilateral. For each of the three noncyclic orders there are two squares of opposite sense.*

When there exist two inscribed squares with the same sense for a given order, as for $l_1l_2l_3l_4$, then l_4' coincides with l_4, and there exists an infinite number of solutions.

THEOREM 7. *When for a given order of a quadrilateral there exist two inscribed squares with the same sense, then there exist an infinite number of inscribed squares for the same order.*

Thus a problem of old standing is solved in a complete* and very simple manner by preparing the way in the complex field.

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* Carnot in his trigonometric solution of the problem (*Géométrie de Position*, p. 374) stated the possibility of three solutions only.

NOTE ON SOME RESULTS CONCERNING FERMAT'S LAST THEOREM

BY H. S. VANDIVER

Let us suppose that the equation

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in integers prime to each other and that p is an odd prime. We shall divide the discussion into two parts. We shall refer to the case where xyz is prime to p as Case I, and to the case where $xyz \equiv 0 \pmod{p}$ as Case II. The present note will be confined to statements of new results obtained by the writer for both cases, without proofs.

1. *Case* I. *Transformations of the Kummer Criteria.* If

$$f_a(t) = \sum_{r=1}^{p-1} r^{a-1} t^r,$$

then it is known that, if (1) is satisfied in Case I, we have

$$\begin{cases} f_{p-1}(t) \equiv 0 \pmod{p}, \\ B_n f_{p-2n}(t) \equiv 0 \pmod{p}, \end{cases} \tag{2}$$

$n = 1, 2, \cdots, \mu - 1$, and $-t \equiv x/y,\ y/x,\ z/x,\ x/z,\ y/z,\ z/y$, $\pmod{p}$, $\mu = (p-1)/2$, where the B's are the Bernoulli numbers, $B_1 = 1/6$, $B_2 = 1/30$, etc. These congruences are known as the Kummer criteria. Transformations of them have yielded the following conditions for Case I:

$$\left(\sum_{r=0}^{p-1} (2r+1)^i t^r\right)\left(\sum_{r=0}^{p-1} (2r+1)^{p-2-i} t^r\right) \equiv 0 \pmod{p},$$

$$(i = 1, 2, \cdots, \mu - 1),$$

$$f_p(t_1) \equiv 0 \pmod{p^2},$$

$$-t_1 \equiv x/y,\ y/x,\ x/z,\ z/x,\ y/z,\ z/y \pmod{p^2}.$$

If e is the smallest integer such that $B_e \equiv 0 \pmod{p}$, we have

$$B_c f_{2k-1}(-t) \equiv 0 \pmod{p}, \tag{3}$$

$$(c = (p - 2k + 1)/2, \quad k = 2, 3, \cdots, e - 1).$$

Kummer and Mirimanoff* have shown that if (1) is satisfied in Case I, then

$$B_{\mu-1} \equiv B_{\mu-2} \equiv B_{\mu-3} \equiv B_{\mu-4} \equiv 0 \pmod{p}.$$

The relation (3) gives material for extension of these criteria.

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* JOURNAL FÜR MATHEMATIK, vol. 128 (1905), pp. 45–68.

2. *Results from a New Method.* In efforts to supplement the relations (2) by independent conditions in Case I, the writer, by the use of new developments in the theory of cyclotomic fields, has derived the following criteria* for Case I:

$$B_s \equiv 0 \pmod{p^2}, \tag{4}$$

$$s = (hp+1)/2, \qquad h = p-4,\, p-6,\, p-8,\, p-10.$$

If $e = a^f$, *where* a *is any prime and* f *any integer, then*

$$\frac{(t^e-1)^p - t^{ep} + 1}{p} \equiv 0 \pmod{p}. \tag{5}$$

This relation possibly also holds if e is any integer, but the formulas which it seems necessary to examine in the general case are very complicated. The condition (5) gives the result: *If* t *belongs to the exponent* n, *modulo* p, *and* n *is the power of a prime, then*

$$1 + \frac{1}{2} + \cdots + \frac{1}{j\nu} \equiv 0 \pmod{p},$$

$$(\nu = (p-1)/n, \qquad j = 1, 2, \cdots, n-1).$$

Hence

$$q(n) \equiv \frac{n^{p-1}-1}{p} \equiv 0 \pmod{p}.$$

In addition, these methods have yielded a new derivation of the known criteria $q(2) \equiv q(3) \equiv 0 \pmod{p}$ without use of (2) or of the criterion of Furtwängler.†

3. *Criteria derived by a second method.* Using transformations of some known theorems regarding cyclotomic fields, the following conditions have been found for Case I.

Let the principal ideal $(\omega(\alpha))$ *be the pth power of any ideal in the field defined by* $\alpha = e^{2i\pi/p}$ *which is prime to* z *and* p; *then*

$$f_{p-n}(s)\left[\frac{d^n \log \omega(e^v)}{dv^n}\right]_{v=0} \equiv 0 \pmod{p},$$

where $s \equiv -x/y \pmod{p}$, $n = 1, 2, \cdots, p-2$, *and* e *is the Napierian base.*

If we have two solutions of (1), say

$$x_1^p + y_1^p + z_1^p = 0,$$
$$x_2^p + y_2^p + z_2^p = 0,$$

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* The first was previously stated in this BULLETIN, vol. 24 (1918), p. 472.

† WIENER BERICHTE, vol. 121 (1912), p. 589.

where z_1 is prime to z_2, then we have as a special case of the above,

$$f_{p-n}(S_1)f_n(S_2) \equiv 0 \pmod{p},$$

$$S_1 \equiv -x_1/y_1, \quad S_2 \equiv -x_2/y_2 \pmod{p}, \quad (n = 1, 2, \cdots, p-1),$$

which in turn includes the relations

$$f_{p-n}(t)f_n(1-t) \equiv 0 \pmod{p};$$

and we obtain also, by a modification of the theorem,

$$f_{p-n}(t)f_n(t) \equiv 0 \pmod{p},$$

n ranging as before, the latter being a known transformation of (2).

4. *Case* II. Under the following assumptions:

1. *None of the Bernoulli numbers*

$$B_k, \quad k = (sp+1)/2, \qquad (s = 1, 3, \cdots, p-4),$$

are divisible by p^2,

2. *The second factor of the class number of the field* $\Omega(\alpha)$ *is prime to* p,

I have proved that (1) is impossible in Case II. A special case of this gives the criterion that if (1) is satisfied in Case II, then the class number of $\Omega(\alpha)$ is divisible by p^2, which in turn may be shown to include the results which Kummer essayed to prove in his 1857 memoir on the last theorem. Taking into account the criteria (4) for Case I, we may cover both cases by the following statement.

Under assumptions 1 *and* 2, *the relation* (1) *is not satisfied in integers* x, y, z, *none zero, for a prime* $p > 2$.

It will be noted that this criterion may be tested for any prime in a finite number of steps.

Kummer* computed the value of the first factor of th eclass number of $\Omega(\alpha)$ for all primes less than 167. Assuming these computations correct, it may be shown that assumption 1 holds for all these primes. According to Kummer† the second assumption holds for all primes less than 100, and it has not been tested for larger primes.

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* BERLINER MONATSBERICHTE, 1874.

† BERLINER ABHANDLUNGEN, 1857, p. 73, verification of second assumption.

CONVEX DISTRIBUTION OF THE ZEROS OF STURM-LIOUVILLE FUNCTIONS*

BY EINAR HILLE

Introduction. In an investigation concerning the distribution in the complex plane of the zeros of the functions of the parabolic cylinder the author of the present note was confronted with the following problem. Given a linear differential equation of the second order of the type

$$w'' + G(z)w = 0, \tag{1}$$

where $G(z)$ is analytic in the region under consideration, *how does the nature of the argument of* $G(z)$ *affect the distribution of the zeros of a given solution* $w(z)$ *of* (1)?

We have some results which throw light on this general question. Roughly speaking, one may say that the argument of $G(z)$ affects the *orientation* of the zeros in the complex plane, whereas the absolute value of $G(z)$ seems to affect the *density* of the distribution. As the problem and the results obtained seem to have some intrinsic value aside from their bearing on the special problem, mentioned above, their separate publication may perhaps be justified.

1. *Bounded Argument of* $G(z)$. Let us consider a convex region B in the z-plane in which $G(z)$ is single-valued and analytic and in which, furthermore, the argument of $G(z)$ is restricted as follows:

$$\vartheta_1 \geqq \arg G(z) \geqq \vartheta_2 \quad (\pi > \vartheta_1 - \vartheta_2 \geqq 0). \tag{2}$$

Let us construct a set of polygonal lines (p) in B in the following manner. Take any point P_0 in the interior of B. Draw from this point all rays which form an angle θ_{P_0} with the positive real axis such that

$$-\tfrac{1}{2}\vartheta_1 < \theta_{P_0} < -\tfrac{1}{2}\vartheta_2. \tag{3}$$

Take a second point P_1 in B on any of these rays and use it as vertex for a second set of rays with the same limitation on the slope-angle θ_{P_1} as on θ_{P_0}. Then choose a third point P_2 and so on. The polygonal line so obtained forms an element p of our set (p). Repeat this construction for all interior points P_0 in B and for every possible choice of points $P_1, P_2, \cdots, P_n, \cdots$ on the resulting rays. The set of all these polygonal lines is the set (p). By the construction we define a positive direction on the line p, namely from P_n to P_{n+1} for all values of n; we always assume the line to be traced in this direction. Now take a solution $w(z)$ of (1) and put $W(z) = w(z)(dw/dz)$. Mark the zeros of $W(z)$ in B. Then we have the following theorem.

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* Presented to the Society, Dec. 29, 1921.

THEOREM I. *There is at least one polygonal line p_0 in the set (p), defined above, which joins all the zeros of $W(z)$ in B. The line p_0 is unique when and only when $G(z) = const.$*

The theorem is equivalent to the following one.

THEOREM II. *If a and b are two zeros of $W(z)$ in B, then*

$$-\tfrac{1}{2}\vartheta_1 + k\pi \leqq \arg (a - b) \leqq -\tfrac{1}{2}\vartheta_2 + k\pi,$$

where k is 0 or 1.

We have proved theorem II in another place* but the proof will be repeated for the benefit of the reader. Multiply equation (1) by $\bar{w}$, the conjugate of w, and integrate between a and b. After an integration by parts we obtain

$$\text{(4)} \qquad \left[\bar{w}\frac{dw}{dz}\right]_a^b - \int_a^b \left|\frac{dw}{dz}\right|^2 \overline{dz} + \int_a^b |w|^2 G(z)dz = 0.$$

But $W(z) = 0$ for $z = a$ and b, hence the integrated part disappears. Putting $z = a + re^{i\theta}$ where $\theta = \arg (b - a)$, we get

$$\text{(5)} \quad e^{2i\theta}\int_0^{r_1} |w|^2 G(z)dr = \int_0^{r_1} \left|\frac{dw}{dz}\right|^2 dr, \quad (r_1 = |b - a|).$$

Thus, denoting the integral on the left-hand side in (5) by $\mathfrak{J}(a, b)$ we observe that $e^{2i\theta}\mathfrak{J}(a, b)$ is a real positive number.

Hence

$$\text{(6)} \qquad 2\theta = 2k\pi - \arg \mathfrak{J}(a, b).$$

But by (2) we conclude that $\vartheta_1 \geqq \arg \mathfrak{J}(a, b) \geqq \vartheta_2$, so that

$$\text{(7)} \qquad k\pi - \tfrac{1}{2}\vartheta_1 \leqq \theta \geqq k\pi - \tfrac{1}{2}\vartheta_2,$$

where the signs of equality hold only when $\vartheta_1 = \vartheta_2$, which clearly corresponds to $G(z) = \text{const.}$, when all the zeros lie on the same straight line.

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* *On the zeros of Sturm-Liouville functions*, theorem XI, ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK, vol. 16, nr. 17, 1921.

2. *Monotonic Variation of* arg $G(z)$. The line p_0 is not unique but the sub-set (p_0) of lines in (p) which pass through all the zeros of $W(z)$ in B meet these points in one and the same order. Suppose we enumerate the zeros $a_1, a_2, \cdots, a_n$ in the order in which they occur on any of these lines. Let ϖ_0 stand for the polygonal line in (p_0) for which $P_\nu = a_{\nu+1}$ $(\nu = 0, 1, \cdots, n-1)$. We are going to show that if the argument of $G(z)$ varies in a monotonic manner along all lines of (p), then ϖ_0 together with the line (a_n, a_1) encloses a convex polygon. This is more precisely stated as follows.

THEOREM III. *If* arg $G(z)$ *is never increasing* (never decreasing) *when* z *traces any line of* (p), *then the angles* arg $(a_{\nu+1} - a_\nu)$ *form a never decreasing* (never increasing) *set.*

To fix our ideas, let us assume arg $G(z)$ never increasing and put $\arg(a_{\nu+1} - a_\nu) = \theta_\nu$. Then we have to prove that $\theta_\nu \leqq \theta_{\nu+1}$, for all values of ν. By (6) we have

$$(8) \qquad \begin{cases} 2\theta_\nu = -\arg \mathfrak{J}(a_\nu\ , a_{\nu+1}), \\ 2\theta_{\nu+1} = -\arg \mathfrak{J}(a_{\nu+1}, a_{\nu+2}). \end{cases}$$

But arg $G(z)$ is never increasing along ϖ_0, so that

$$(9) \qquad \begin{cases} \arg G(a_{\nu+1}) \leqq \arg \mathfrak{J}(a_\nu\ , a_{\nu+1}) \leqq \arg G(a_\nu), \\ \arg G(a_{\nu+2}) \leqq \arg \mathfrak{J}(a_{\nu+1}, a_{\nu+2}) \leqq \arg G(a_{\nu+1}). \end{cases}$$

In view of (8) and (9) we conclude that $\theta_\nu \leqq \theta_{\nu+1}$ as asserted.

3. *A Zero-free Region.* Formula (5) suggests a still more general point of view on the problem in hand. At each point z of the plane where $G(z)$ is analytic and $\neq 0$ there is a critical direction θ_z determined by

$$(10) \qquad \theta_z = -\tfrac{1}{2} \arg G(z).$$

z and θ_z define a lineal element p_z. These elements envelop a set of curves $\mathfrak{S}$ which are integral curves of the differential equation

$$(11) \qquad \mathfrak{Im}[\sqrt{G(z)}\, dz] = 0.$$

In fact, from (10) it follows that at the point z

$$(12) \qquad \tfrac{1}{2} \arg G(z) + \arg dz \equiv 0 \pmod{\pi},$$

which is equivalent to (11).

Let us take a point z_1 where $G(z)$ is analytic and $\neq 0$. Mark the singular points $a_1, a_2, \cdots, a_n, \cdots$ of $G(z)$ and join them with ∞ by means of straight lines which produced backwards pass through z_1. These lines $L_1, L_2, \cdots, L_n, \cdots$ we consider as cuts in the plane.

Take a pencil of rays $l(\varphi, z_1)$ with $z = z_1$ as vertex. Let us follow a special ray $l(\varphi_0, z_1)$ from z_1. Note the lineal elements along l; if $\varphi_0 \neq \theta_{z_1}$, we proceed until we arrive at a point $z_2 = z_2(\varphi_0)$ where $\theta_{z_2} \equiv \varphi_0 \pmod{\pi}$; and the remainder of the ray we leave out. From the rays that pass through the singular points we leave out at least the lines L_ν. The rays $\varphi_0 \equiv \theta_{z_1} \pmod{\pi}$ we follow until we meet a second point z_2 for which $\theta_{z_2} \equiv \varphi_0$. The points $z_2(\varphi)$ form a certain curve which may lie partly at infinity and in general also has isolated discontinuities. The finite points, except possibly the singular points a_ν, form part of an analytic curve $\mathfrak{T}(z_1)$ which is the locus of the points of tangency of the pencil of straight lines through $z = z_1$ with the family of curves $\mathfrak{S}$. From the differential equation of the pencil and of the family, namely,

$$\left\{\begin{array}{ll}\mathfrak{Im}[\sqrt{G(z)}\,dz] & = 0,\\ \mathfrak{Im}[dz/(z - z_1)] & = 0,\end{array}\right. \tag{13}$$

we conclude that the equation of $\mathfrak{T}(z_1)$ is

$$\mathfrak{Im}[\sqrt{G(z)}\,(z - z_1)] = 0. \tag{14}$$

If $z_2(\varphi)$ is discontinuous at $\varphi = \varphi_1$, we assume the existence of $z_2^- = \lim z_2(\varphi_1 - \epsilon)$ and of $z_2^+ = \lim z_2(\varphi_1 + \epsilon)$, neither of which is required to be finite. If we add the part of the ray $l(\varphi_1)$ from z_2^- to z_2^+, we make the boundary curve continuous at $\varphi = \varphi_1$. If $z_2(\varphi_\nu) = a_\nu$, one of the singular points, we have to complete $z_2(\varphi)$ by parts of the edges of the cut L_ν, namely, from a_ν to z_2^- and from a_ν to z_2^+ respectively. What we add from the cuts we consider as *inaccessible* parts of the boundary, whereas the additions from the interior of the cut plane as well as the part of $\mathfrak{T}(z_1)$ which belongs to $z_2(\varphi)$ are regarded as accessible.

The points on $l(\varphi, z)$ from z_1 to $z_2(\varphi)$ $(0 \leqq \varphi < 2\pi)$ form a region $*(z_1)$ which we shall call the *star* of z_1.* We count the accessible part of the boundary as belonging to the star;

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* In a paper *Oscillation theorems in the complex domain*, to appear in the TRANSACTIONS OF THIS SOCIETY, we have given a different definition of the star. The definition given above is simpler and slightly more general.

the inaccessible points on the cuts and the *vertex* $z = z_1$ we do not include in $*(z_1)$. Then we have

THEOREM IV. *If* $W(z) = w(z)(dw/dz)$, *where* $w(z)$ *is a solution of* (1), *and if* $W(z_1) = 0$, *then* $W(z)$ *does not vanish in the star of* z_1.

The proof is obvious. Suppose z_3 is a point in $*(z_1)$. Then

$$(15) \quad (k+1)\pi > 2 \arg (z - z_1) + \arg G(z) > k\pi, \; (k{=}0 \text{ or } 1),$$

for all values of z on the segment (z_1, z_3) and where k only depends upon z_3. This shows that the integral on the left-hand side in formula (5) with $r_1 = |z_3 - z_1|$ cannot be positive, which is necessary if $z = z_3$ is to be a zero of $W(z)$.

If $\arg G(z)$ satisfies the restriction (2) in a convex region B, then θ_z satisfies the inequality (3); that is to say, the line part of the lineal element p_z is a line in (p) as far as the neighborhood of z is concerned. If the variation of the argument of $G(z)$ along lines in (p) is monotonic, then the parts of the curves $\mathfrak{S}$ which lie in B are all curved in the same way. In fact the curvature of a curve $\mathfrak{S}_0$ is found to be

$$(16) \qquad \frac{1}{r} = -\frac{1}{2}\frac{d\phi}{ds},$$

where $\phi = \arg G(z)$ and s is the length of arc along $\mathfrak{S}_0$.* The curve $\mathfrak{S}_0$ lies to the right of the positive tangent if $(d\phi/ds) > 0$, but to the left if $(d\phi/ds) < 0$.

Assume, to fix our ideas, $(d\phi/ds) < 0$. Then the part of $\mathfrak{T}(z_1)$ that lies in B lies to the left of p_{z_1} except at $z = z_1$, which is the point of tangency. $\mathfrak{T}(z_1)$ is convex with regard to p_{z_1} and cuts the boundary of B in two points only. The points on and to the right of p_{z_1} in B belong to $*(z_1)$, and likewise those on and to the left of $\mathfrak{T}(z_1)$.

It is worth noticing that the form of the star depends only upon the argument of $G(z)$. Thus if we replace $G(z)$ by $k^2G(z)$, where k is real, we get exactly the same configuration. The frequency of the oscillation is of course changed by such a replacement.

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* The curvature is by definition $d\theta_z/ds$.

BOOKS ON FOURIER SERIES

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Second edition, revised throughout and enlarged. Vol. 1. Cambridge, the University Press, 1921. xv + 671 pp.

Introduction to the Theory of Fourier's Series and Integrals. By H. S. Carslaw. Second edition, completely revised. London, Macmillan and Company, 1921. xi + 323 pp.

The study of Fourier's series has exercised a profound influence upon the development of the theory of functions of a real variable.* Any one familiar with this influence and with the close relationship, both historical and inherent, between the two theories, will not be surprised that the two treatises mentioned above should have a considerable number of points in common. He will also not be surprised that in each case the plan of writing such a work had its origin in the study of Fourier's series in connection with their applications to problems of mathematical physics. The points of difference in scope and content between the two books are due in the main to the difference in the ultimate aim of the two writers. Professor Carslaw decided to write a book primarily for the worker in applied mathematics who has occasion to make use of Fourier's series and integrals; Professor Hobson elected to meet the needs of the pure mathematician whose work deals directly or indirectly with functions of a real variable.

A casual glance at the table of contents of Carslaw's work, however, will convince the informed reader that the needs of the pure mathematician of one generation are very apt to become the needs of the applied mathematician of the next generation. It is not so long ago that it would have been considered very unorthodox to include such topics as a discussion of Dedekind's theory of irrational numbers, the nature of uniform and non-uniform convergence, and a fairly complete treatment of the Riemann integral in a book intended for the applied mathematician. So, if the reviewer admits that Hobson's book is mainly for the pure mathematician, he does it with the mental reservation that the statement applies only to the present time and the immediate future. He does not agree with the implications contained in Hobson's statement on page 432 that the Riemann integral "will continue to be the basis upon which the practical applications of the Integral Calculus rest", but thinks it quite likely that at some future time, more or less distant, certain workers in applied mathematics may find their center of interest transferred from the Riemann integral to the Lebesgue integral, or some other integral still more general, just as at the present time some of those pursuing applied mathematics have found it desirable to change from a basis of euclidean geometry to a basis of non-euclidean geometry.

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* Cf. E. B. Van Vleck, *The influence of Fourier's series upon the development of mathematics*, SCIENCE, new ser., vol. 39 (1914).

The rapidity with which the Lebesgue integral has become the standard type in present-day analysis is well exhibited by the difference in the relative emphasis placed on the Riemann and Lebesgue theories in the two editions of Hobson's book. In the first edition the latter theory was accorded a position of secondary importance; in the present edition it is placed on an equal basis, or if anything receives the greater stress. It is also illuminating in this connection to note that most of the new material included in the present edition is connected with advances concerned with or dependent on Lebesgue's theory.

As the reader will infer from the preceding paragraph, the second edition of Hobson's work takes due account of the great developments that have taken place during the past fifteen years in the theory of functions of a real variable and particularly in the theory of integration. Likewise Carslaw has not been content to rest on the laurels of his first edition, but has incorporated in his second edition many of the recent advances in the theory of Fourier's series and integrals that seem of special significance to the worker in applied mathematics. Thus both authors have aimed in the second edition as in the first to write the best possible book of the type selected on the basis of all the literature available at the time of writing. In both cases they have approached measurably near to the high ideal that they set for themselves.

Since the first editions of the two books are so widely known, the present review will not consider in great detail such portions of the second editions as are carried over from the first with relatively little change, but will concern itself mainly with the additions and the more important revisions that have been made.

Turning first to Hobson's work, we find that Chapter I is devoted to an adequate discussion of the system of real numbers, presenting both the Dedekind and the Cantor treatment of the irrationals. It is practically unchanged from the first edition except for the addition of a treatment of mathematical induction due to Padoa. The purpose of this discussion is to show that the principle referred to is a theorem which may be deduced from the properties of a simply infinite ascending aggregate by means of the principle of contradiction. This is essentially a refutation of Poincaré's contention that the principle of mathematical induction is a special characteristic of mathematical reasoning and cannot be reduced to the principle of contradiction.

Chapters II and III, which deal respectively with the descriptive properties and the metric properties of sets of points, constitute the revised form of the second chapter of the first edition. The material contained in this chapter has been much expanded in order to prepare the way for the more extensive treatment of Lebesgue integrals and their generalizations, and in numerous instances the form of the exposition has been considerably changed. One of the most extensive alterations of this sort is in connection with the treatment of non-linear sets of points. In the first edition this treatment was postponed until the exposition of the properties of linear sets had been completed. In the present edition the notion of sets of points in two or more dimensions is introduced early in Chapter II, and the more important theorems with regard to linear sets are extended at once to sets of several dimensions after they have been proved for the case of one dimension. This change has both advantages and disadvantages. It does serve to focus the attention on the generality of the theorems involved, but on the other hand it interrupts the continuity of the argument. Opinions will probably differ as to whether or not it constitutes an improvement.

Chapter IV, which deals with transfinite numbers and order types, corresponds to Chapter III in the first edition. Relatively little change has been made in the material presented, except that the discussion of certain disputed points in the theory of assemblages, which concludes the chapter, has been revised and expanded.

Chapter V, which deals with general properties of functions of a real variable and which corresponds to Chapter IV of the first edition, contains a considerable amount of new material. This is quite natural in view of the many new developments in function theory proper during the period from 1907 to 1920. Some of this added material was available in 1907, but in most of these cases it was probably the light shed by later developments which changed the author's opinion as to its relative importance. Among the new topics discussed may be mentioned in particular the following: absolute continuity, approximate continuity, the symmetry of functional limits, functions of a variable aggregate. In connection with the latter topic, the notion of a general variable is briefly described and some of the recent developments centering about this idea are mentioned. In view of the prominent rôle of the general variable in much recent work in analysis of fundamental importance, a fuller account of it would seem desirable, even in a work devoted primarily to a very complete treatment of classical analysis.

Chapters VI, VII, and VIII are devoted to the theory of integration, and correspond to Chapter V and a small portion of Chapter VI of the first edition. This part of the work has had the greatest expansion, and this expansion is mostly in connection with the Lebesgue integral and some of its generalizations. As pointed out before, one of the chief characteristics of the present edition of the book, as contrasted with the first, is the shift in emphasis from the Riemann theory to the Lebesgue theory. The tendency in the later edition is to make this latter theory the central topic in integration and group the other developments about it. Thus, although the Riemann integral, on account of its historical priority and the somewhat greater concreteness of the geometric ideas underlying its definition, is naturally treated first, some of the important properties of the definite integral, such as the mean value theorems, are proved for the first time after the introduction of Lebesgue integrals.

In addition to a complete account of the Riemann and Lebesgue integrals, the three chapters devoted to integration contain more or less extended treatments of other types of integrals, such as the Riemann-Stieltjes integral, the Lebesgue-Stieltjes integral, Hellinger's integrals, the Denjoy integral, and the Young integral. Since the theory of integration is still in a stage of active development, it is rather too much to expect a treatise such as Professor's Hobson's, which naturally aims to include only such researches as have reached a state of comparative completeness, to give an entirely adequate picture of the present state of the theory. Moreover, some authorities might disagree with him as to the relative emphasis he accords to the various generalizations of the notion of integration. But the reader who has perused with care the 230 odd pages devoted to this highly important topic will have a fairly satisfactory idea of the wide extension of its scope involved in the newer definitions and the much greater power of these definitions in dealing with the applications of the theory.

The reviewer has earlier expressed his high opinion of the book as a whole. It is hardly to be expected that a work of this size and scope should be free from minor defects; there are certain of these that it seems worth while to indicate. Hobson's references to the literature would have been much more valuable to the reader if he had uniformly included dates of publication in the case of all important memoirs. The almost uniform omission of such dates is hardly excusable in an account of a theory where the works of various writers have so many interrelations and the reader naturally wishes to orient himself as to the order of ideas and the reaction of one line of thought upon another. The use of the symbol $\sim$ to indicate approach to a limit also seems unwise to the reviewer. While this symbol has had some connection with approach to a limit when used in the case of asymptotic series, it has generally been used to represent correspondence in a broader sense. The symbol $\rightarrow$ is much more suggestive of the underlying notions and has been widely adopted by analysts in the time that has elapsed since its introduction.* It does not appear likely that Hobson's notation will supplant it.

Turning now to Carslaw's book, we find that the first volume of the second edition exhibits an expansion over the corresponding material in the first edition that is even greater proportionally than in the case of Hobson's book. This will not be surprising to those familiar with recent developments in the theory of Fourier's series and integrals, for many of these developments have been of special significance to those primarily interested in the applications of the theory. The new material added by Carslaw is not exclusively composed of such new developments, however. A considerable portion of it is devoted to a more elaborate development of certain topics of pure analysis that are of fundamental importance for those who seek to use in any rational manner such tools as Fourier's series and integrals in the field of applied mathematics.

The book begins as before with an interesting historical introduction, which in the present edition has been somewhat revised and expanded. Chapter I, devoted to an exposition of the system of real numbers, and Chapter II, dealing with the more fundamental properties of infinite sequences and series, contain substantially the same material as in the first edition, though they have been considerably revised. Chapter III of the present edition is new and contains a development of the notions of function and limit and some of the more fundamental properties of functions of one or more variables. The addition of this chapter is in line with a suggestion by the late J. E. Wright in the course of his review of the first edition in this BULLETIN.

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* It may be worthy of note in this connection that this BULLETIN has adopted this notation from the beginning of the present volume. (The editors.)

Chapter IV deals with the same material as the first part of Chapter IV in the first edition, but the material has been much expanded and revised. An adequate treatment of the Riemann integral and its more important properties has been included in this chapter. Chapter V is the original Chapter III in revised form and with a considerable amount of new material at the end. This includes a proof of Bromwich's theorem on convergence factors which is of importance in many applications of Fourier's series and allied developments, and a discussion of the integration of infinite series. Chapter VI contains the remaining portion of Chapter IV, namely, the treatment of definite integrals involving a parameter. This portion also has been revised and expanded.

Chapter VII corresponds to the original Chapter V and is devoted to the discussion of the more fundamental theorems with regard to the convergence and summability (C1) of the Fourier development of an arbitrary function. Among the new material added may be mentioned the proof of the convergence of Fourier's series by means of Fejér's theorem on the summability (C1) and a general theorem with regard to series summable (C1) due to G. H. Hardy; also a discussion of Poisson's integral.

Chapter VIII, dealing mainly with the uniform convergence of Fourier's series and the validity of term by term differentiation and integration of such series, and Chapter IX, devoted to the consideration of the approximation curves and Gibb's phenomenon, correspond to Chapters VI and VII of the original edition. The material on Gibb's phenomenon is new and is founded mainly on Bôcher's discussion in the ANNALS OF MATHEMATICS and the further treatment of the subject by the author in the AMERICAN JOURNAL OF MATHEMATICS.

Chapter X, which deals with Fourier's integrals, corresponds to the original Chapter VIII. In its present form it includes recent extensions of the theory of these integrals due to Pringsheim. Appendix I, which is new, gives a brief account of certain methods of harmonic analysis that are useful when the function to be developed is not given in analytic form but only by means of a curve obtained from observations. Appendix II contains the bibliography of Appendix A in the first edition, brought up to date. This bringing up to date involves the addition of seven and a half pages of titles for the period 1906–1920.

CHARLES N. MOORE

SHORTER NOTICES

Oeuvres de G. H. Halphen. Publiées par les soins de C. Jordan, H. Poincaré, É. Picard, avec la collaboration de E. Vessiot. Tome III.* Paris, Gauthier-Villars, 1921. xii + 518 pp.

The Paris Academy of Sciences proposed as the subject of the grand prize in mathematics for 1880 the following question: "To perfect in some important part the theory of ordinary linear differential equations." Halphen submitted a memoir in which he proposed to determine those ordinary linear differential equations which are reducible to integrable forms by means of transformations of a certain class; and the prize was awarded to this memoir. The text of Hermite's report on the memoir is reproduced in an article on Halphen published by C. Jordan in LIOUVILLE'S JOURNAL in 1889 and reprinted on pages v to xii of the volume under review. (See also p. 33 of vol. I of Halphen's *Oeuvres.*)

Halphen formulated in the following manner the exact "double question" which he treated:

(1) Having given a linear differential equation with variable X and unknown Y, to determine if there exists a substitution

$$x = \varphi(X), \qquad y = Y\psi(X),$$

such that, x and y being taken for the new variable and the new unknown, the transformed equation belongs to one of the following three categories:

I. Equations with constant coefficients;
II. Equations whose general integral is rational;
III. Equations with doubly periodic coefficients with the same periods and with a uniform general integral.

(2) Having determined the existence of the substitution, to effect the integration.

The memoir in which he treats this question is reprinted on pages 1–260 of the volume under review. It opens with an excellent fourteen-page summary of the results attained. The response to the first question is entirely decisive: necessary and sufficient conditions are obtained (and summarized in the introduction) for the existence of the substitution named of such sort that one may apply them to a particular equation by means of straightforward reckoning. The propositions relating to the second part are not quite so definitive; but they are far-reaching enough to have important applications, particularly to equations of the second order.

The judgment of the Academy, awarding the grand prize to Halphen for this memoir, was scarcely rendered when the Berlin Academy proposed for the Steiner prize "the solution of an important question in the theory of twisted algebraic curves." In 1870 Halphen had published a short note on the classification of these curves. He now returned to this subject and produced his classic memoir on the classification of twisted algebraic curves (reprinted on pp. 261–455 of the volume under review). It was crowned conjointly with a memoir by Noether submitted on the same occasion. By many this is esteemed to be Halphen's chief work. Some account of it is given in the review of volumes I and II in connection with Poincaré's analysis of the memoir on pages xxx–xxxi of volume I.

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* For a short review of volumes I and II see this BULLETIN, vol. 27 (1920–21), pp. 466–468.

Besides these two master memoirs, volume III of the *Oeuvres* contains the introductory article by C. Jordan already referred to, and two articles by Halphen closely related to the former of the two already treated. They both deal with linear differential equations of the fourth order: one is a brief note (pp. 457–461) from the COMPTES RENDUS and the other is a longer article (pp. 463–514) from the ACTA MATHEMATICA on the invariants of linear differential equations of the fourth order.

The four articles reprinted in this volume may justly be said to contain two great contributions by Halphen. The one is the memoir on twisted algebraic curves and the other is the work on linear differential equations contained in the first, third and fourth papers. These are all entirely inspired by the theory of differential invariants (to which Halphen had previously made characteristic contributions); and they are treated together by Halphen himself (vol. I, pp. 32–35) in the summary of his work prepared by him on the occasion of his candidacy before the Paris Academy in 1885.

R. D. CARMICHAEL

Graphical and Mechanical Computation. By Joseph Lipka. New York, John Wiley and Sons, 1918. ix + 264 pp.

This book is something new in American text-books. It treats two subjects of vital importance to engineering students that heretofore have found little place in the curricula of American engineering schools. The first part of the book is devoted to the graphic solution of engineering problems by means of networks of scales, various kinds of coordinate paper, the charting of equations in three variables and more particularly by means of alignment charts. The second half of the book is concerned with the question of empirical formulas for both non-periodic and periodic curves giving numerical, graphic and mechanical methods for determining the constants. For data which cannot be fitted to convenient formulas, numerical, graphic and mechanical methods are developed in the last two chapters for interpolation, differentiation and integration.

The book is rich in illustrations and applications to practical engineering problems, showing clearly the value of the graphic methods in reducing the drudgery of long computations. The three chapters on alignment charts are recommended to the student who wishes to know something of the nomographic charts of Professor M. d'Ocagne. The work is also published in two volumes.

A. R. CRATHORNE

Funktionentheorie. By Konrad Knopp. 2d edition. Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger. Part I, 1918, 140 pp. Part II, 1920, 138 pp.

This "Funktionentheorie" is a completely revised edition of the work that appeared under the same title in 1913 in the Sammlung Göschen. It is issued in a form similar to the old one by the Vereinigung wissenschaftlicher Verleger which, since the war, has continued the publishing activities of several houses, among them the Göschen'sche Verlagshandlung.

The book, grouping its material around the centrally important concept of the monogenic analytic function, affords a very satisfactory brief introduction to the study of the theory of functions of a complex variable. By maintaining throughout sufficient emphasis upon the central concept, and by connecting various parts of the theory by their relation to this concept, a balance that is very desirable in a book of this kind is secured between the more and the less theoretical aspects of the subject. After a chapter on point sets, *Wege* (i.e., curves consisting of a finite number of rectifiable pieces), etc., there are taken up the integral theorems, development in series, analytic continuation and singular points. Then follow, in the second volume, short sections on entire functions, meromorphic functions and periodic functions (with brief mention of the elliptic functions); and finally a section on multiple-valued functions.

The book is clearly written, well printed, and practically free from misprints. The Cauchy integral theorem is proved for a closed *Weg* by means of approximating polygonal lines and by reducing the theorem ultimately to the case in which the area enclosed by the curve is a triangle, in a manner similar to that followed by Jordan in volume I of his *Cours d'Analyse.* A number of well-chosen exercises are distributed throughout the book.

ARNOLD DRESDEN

Réflexions sur la Métaphysique du Calcul Infinitésimal. By Lazare Carnot. I and II. Paris, Gauthier-Villars, 1921.

It is not generally known that this treatise was translated into English by William Dickson and published, with notes, in the PHILOSOPHICAL MAGAZINE of London, volumes 9 and 10, for the years 1800 and 1801. Maurice Solovine, the editor of the present edition, is mistaken when he states that J. K. Hauff's German translation (1800) and G. B. Magistrini's Italian translation (1803) antedated translation into English. To be sure, a later English version, prepared by William Robert Browell, did appear at Oxford in 1832. The various translations bear testimony to the high esteem in which the booklet was held by Carnot's contemporaries. Carnot makes no reference to English mathematicians of the eighteenth century who came after Newton; he was unacquainted with Bishop Berkeley's onslaught as found in the ANALYST. Berkeley and Carnot exerted a strong influence on the development of the philosophy of the calculus: the one by his destructive criticism, the other by his constructive efforts. Carnot lays great stress on the doctrine of the compensation of errors in infinitesimal analysis, being unaware that before him this view-point had been presented by Berkeley and others. While D'Alembert and Lhuilier were partial to the method of limits, Carnot found that method a tortuous foot-path in which it was difficult to avoid being bewildered. Mathematicians interested in the history and philosophy of the calculus will welcome this reprint of the greatly enlarged edition of 1813. It appears in the series of inexpensive editions of *Les Maîtres de la Pensée Scientifique.*

FLORIAN CAJORI

NOTES

On account of certain regulations of the United States Postoffice Department, the April–May number of this BULLETIN was given only one serial number, though it is in fact a double number. It is number 4 of volume 28, and number 304 in the total numbering. For this reason, the present volume will contain only nine numbers according to this serial numbering, but there will be issues for each of ten months, as usual, and the total number of pages in the volume will remain at approximately 500 pages, as usual. The only irregularity will be in the serial number of the April–May issue, as explained above.

On April 18, a celebration in honor of Professor Charlotte Angas Scott was held at Bryn Mawr College by her former students. (See this BULLETIN, vol. 28, p. 72.) In the afternoon, after an address of welcome by President M. Carey Thomas and an appreciation of Professor Scott as a teacher and an investigator by Miss Marion Reilly, Professor A. N. Whitehead gave an address on *Some principles of physical science.* The lecture was followed by a tea for all the guests at the Deanery, the home of President Thomas. At a dinner in the evening, there were present about seventy members of the American Mathematical Society, about seventy former students, and a number of members of the Faculty of the College. Representatives of each of these groups gave short talks testifying to Professor Scott's early brilliant record in England, to the inspiration and encouragement that her research and career have brought especially to women beginning graduate work in mathematics, and to the invaluable services that she renders to the College by her teaching, her committee work, and her council.

The October number (vol. 43, No. 4) of the AMERICAN JOURNAL OF MATHEMATICS contains the following papers: *On some properties of general manifolds relating to Einstein's theory of gravitation*, by J. A. Schouten and D. J. Struik; *Geometrical theorems on Einstein's cosmological equations*, by Edward Kasner; *On the Fermat and Hessian points for the non-euclidean triangle and their analogues for the tetrahedron*, by C. M. Sparrow; *The Cauchy-Lipschitz method for infinite systems of differential equations*, by W. L. Hart; *Boundary value and expansion problems; formulation of various transcendental problems*, by R. D. Carmichael; *Reciprocity in a problem of relative maxima and minima*, by J. K. Whittemore.

Professor Gino Loria announces that the BOLLETTINO DI BIBLIOGRAFIA E STORIA DELLE SCIENZE MATHEMATICHE, which he founded and of which he has been editor for many years, will in future be published as an historical and bibliographical section of the BOLLETTINO DI MATEMATICA, of which Dr. A. Conti is editor. It is proposed to retain all the essential features of Professor Loria's BOLLETTINO under the new form.

At the meeting of the National Academy of Sciences held at Washington April 24–26, 1922, the following mathematical papers were read: *Some extensions in the mathematics of hydro-mechanics*, by R. S. Woodward; *Normal coordinates and Einstein space*, by G. D. Birkhoff; *Algebraic solutions of Einstein's cosmological equations*, by Edward Kasner; *The geometry of paths*, by Oswald Veblen. Professor H. A. Lorentz, of the University of Leiden, delivered the evening address on April 24, at the invitation of the Academy and the Carnegie Institution of Washington. Professor L. P. Eisenhart was elected a fellow of the Academy, and Professor Albert Einstein a foreign associate.

The following fourteen doctorates with mathematics as major subject were conferred by American universities in the academic year 1920–21; the title of the dissertation is added in each case: Nina M. Alderton, California, *Involutory quartic transformations in space of four dimensions;* Beulah M. Armstrong, Illinois, *Mathematical induction in group theory;* E. M. Berry, Iowa, *Diffuse reflection,* Rachel Blodgett, Radcliffe, *Determination of the coefficient in interpolation formulæ and a study of the approximate solution of integral equations;* P. H. Daus, California, *Normal ternary fraction expansions for the cube roots of integers;* W. E. Edington, Illinois, *Abstract group definitions and applications;* M. C. Foster, Yale, *Rectilinear congruences referred to special surfaces;* Philip Franklin, Princeton, *Four color problem;* Mayme I. Logsdon, Chicago, *Equivalence and reduction of pairs of hermitian forms;* Irwin Roman, Chicago, *Transformation of waves through a symmetrical optical instrument;* D. V. Steed, California, *Lines on the hypersurface of order* $2n - 3$ *in space of* n *dimensions;* Jung Sun, Syracuse, *Some determinant theorems;* Flora D. Sutton, Johns Hopkins, *Certain chains of theorems in reflective geometry;* F. E. Wood, Chicago, *Certain relations between the projective theory of surfaces and the projective theory of congruences.*

The Carl Zeiss Foundation of Jena announced, at the meeting of mathematicians and physicists held at Jena in September, 1921, the establishment of a prize in memory of Ernst Abbe, "for the advancement of the mathematical and physical sciences and their applications." The interest on a fund of 100,000 marks will be awarded every two years for important research in these fields. The prize will be awarded for work in mathematics in 1924, in physics in 1926, and in applied mathematics and physics in 1928.

Cambridge University has awarded a Smith's prize to E. A. Milne, of Trinity College, for an essay entitled *Studies in the theory of radiative equilibrium.*

Professor René Baire has been elected correspondent of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor M. Noether.

Professor A. S. Eddington has been elected a foreign honorary member of the American Academy of Arts and Sciences.

The following advanced courses in mathematics are announced for the academic year 1922–23:

Columbia University.—By Professor T. S. Fiske: Differential equations (first term).—By Professor F. N. Cole: Algebra.—By Professor D. E. Smith: History of mathematics; Practicum in the history of mathematics.—By Professor C. J. Keyser: Introduction to mathematical philosophy (first term); Logical foundations of mathematics.—By Professor Edward Kasner: Einstein's theory of gravitation.—By Professor W. B. Fite: Theory of functions.—By Professor J. F. Ritt: Functions of several complex variables (first term); Algebraic numbers (second term).—By Dr. G. A. Pfeiffer: Isoperimetric problems (second term).—By Dr. Jesse Douglas: Differential geometry (first term).

Cornell University.—By Professor J. I. Hutchinson: Entire functions. By Professor Virgil Snyder: Algebraic geometry.—By Professor F. R. Sharpe: Vector analysis.—By Professor W. B. Carver: Advanced calculus.—By Professor Arthur Ranum: Differential geometry.—By Professor D. C. Gillespie: The definite integral.—By Professor W. A. Hurwitz: Infinite series.—By Professor C. F. Craig: Probabilities.—By Professor F. W. Owens: Projective geometry.—By Professor H. M. Morse: Einstein's theory (first term); Dynamical systems (second term).—By Professor W. L. G. Williams: Modern higher algebra.—By Dr. F. W. Reed: Elementary differential equations.—By Mr. H. S. Vandiver: Finite groups.—By Dr. G. M. Robison: Advanced analytic geometry.

Harvard University.—All courses meet throughout the year, except those marked with an asterisk, which meet for half a year.—By Professor W. F. Osgood: Differential and integral calculus (advanced course); Theory of functions (introductory course).—By Professor J. L. Coolidge: Probability*; Algebra*; Algebraic plane curves.—By Professor E. V. Huntington: The fundamental concepts of mathematics*.—By Professor O. D. Kellogg: Dynamics (second course); Introduction to the theory of potential functions and Laplace's equation*; Potential functions (advanced course)*.—By Professor G. D. Birkhoff: The analytic theory of heat and problems in elastic vibrations*; Linear differential equations of the second order, real variables*.—By Professor W. C. Graustein: Introduction to modern geometry; Differential geometry of curves and surfaces.—By Dr. J. L. Walsh: Infinite series and products*; Theory of numbers*; Entire functions*.—By Dr. Philip Franklin: Elementary theory of differential equations*; Analysis situs*.—Dr. Walsh and Dr. Franklin will conduct a fortnightly seminar in analysis. Courses of research are offered by Professor Osgood in the theory of functions, by Professor Coolidge in geometry, by Professor Kellogg in the theory of potential functions, by Professor Birkhoff in the theory of differential equations, and by Professor Graustein in geometry.

Professor Alexander Ziwet has presented his private library, consisting of more than five thousand books on mathematics and mechanics, to the University of Michigan. This important collection includes many rare works.

At the University of Strasbourg, Dr. E. Bauer, maître de conférences, has been promoted to a professorship of mathematics.

Mr. W. H. Durfee, of Harvard University, has been appointed assistant professor of mathematics at Hobart College.

Mr. J. A. Northcott, of Columbia University, has been promoted to an assistant professorship of mathematics, in charge of extension courses.

Dr. C. A. Nelson, of Western Reserve University, has been appointed associate in mathematics at Johns Hopkins University.

At the University of Minnesota, Professor A. L. Underhill has leave of absence for the year 1922–23; he expects to spend the year in France.

The following appointments to instructorships in mathematics are announced: Cornell University, Mr. W. W. Elliott; Harvard University, Dr. Philip Franklin (Pierce instructor), Mr. K. W. Holbert, Mr. H. C. Shaub, Mr. M. M. Slotnick, and Mr. D. V. Widder; University of Minnesota, Mr. H. W. Chandler.

Professor D. E. Smith, vice-president of the Society, and Mrs. Smith were seriously injured in an automobile accident on May 30 in Northern France. Both recovered after being in a hospital for over a month. They will return to America in September.

Sir A. B. Kempe, president of the London Mathematical Society in 1894, and for many years treasurer of the Royal Society, died April 27, 1922, at the age of seventy-three years.

Professor J. B. Coit, of Boston University, died July 26, 1921, at the age of seventy-two years.

Dr. G. B. Halsted, professor of mathematics at the University of Texas from 1882 to 1903, and later at St. John's College, Kenyon College, and the Colorado State Teachers' College, died in New York City March 19, 1922, at the age of sixty-nine years. Dr. Halsted had been a member of the American Mathematical Society from the time of its organization, in 1891, as the New York Mathematical Society. He was the author of several books and papers on non-euclidean geometry, and on the foundations of geometry.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

ANDOYER (H.). L'œuvre scientifique de Laplace. Paris, Payot, 1922. 16mo. 162 pp.

BERKHAN (G.). See ENCYKLOPÄDIE.

BIANCHI (L.). Lezioni di geometria differenziale. 3a edizione interamente rifusa ed ampliata, in due volumi. Volume 1. Pisa, Spoerri, 1921. 8vo. 800 pp.

DUPORT (—.). Les transformations de Backlund. Paris, Gauthier-Villars, 1921. 8vo. 10 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 1, Heft 7: G. Berkhan und W. F. Meyer, Neuere Dreiecksgeometrie. H. Rothe, Systeme geometrischer Analyse, 1ter Teil. Leipzig, Teubner, 1921.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 3, Heft 5: E. Salkowski, Dreifach-orthogonale Flächensysteme. Leipzig, Teubner, 1921.

FARKAS (A.). Die Auflösung einiger unbestimmten Gleichungen höheren Grades durch Reihen. Nagyenyed, Aiud, 1920. 7 pp.

GEIRINGER (H.). Die Gedankenwelt der Mathematik. Berlin und Frankfurt, Verlag der Arbeitsgemeinschaft, 1922. 200 pp.

HAMPEL (H.). Ueber Aoustsche Kurven. (Diss., Halle-Wittenberg.) Halle, 1921. 86 pp.

HILBERT (D). Die Grundlagen der Geometrie. 5te Auflage. (Wissenschaft und Hypothese, Nr. 7.) Leipzig, 1922.

JOHNSON (W. E.). Logic. Part 2: Demonstrative inference, deductive and inductive. Cambridge, University Press, 1922. 20 + 258 pp.

KOWALEWSKI (A.). Die Buntordnung. Mathematische, philosophische und technische Betrachtungen über eine neue kombinatorische Idee. Heft 1. Leipzig, Engelmann, 1922. 53 pp.

MEYER (W. F.). See ENCYKLOPÄDIE.

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THE APRIL MEETING OF THE SAN FRANCISCO SECTION

The thirty-ninth regular meeting of the San Francisco Section of the American Mathematical Society was held at Stanford University on Saturday, April 8. Professor R. E. Allardice presided. The total attendance was twenty-one, including the following sixteen members of the Society:

Alderton, Allardice, Bernstein, Blichfeldt, Buck, Cajori, Daus, Haskell, Hoskins, Lehmer, Libby, W. A. Manning, Moreno, F. R. Morris, Noble, Pauline Sperry.

The Secretary of the Section was directed to send a letter of felicitation to the Chicago Section on the occasion of its twenty-fifth anniversary, and a letter of felicitation to Professor Scott on the occasion of the celebration held in her honor at Bryn Mawr College on April 18.

The dates of the next two regular meetings of the Section were fixed as October 21, 1922, University of California, and April 7, 1923, Stanford University.

Titles and abstracts of papers read at this meeting follow below. The papers of Professors Bell and Carpenter were read by title.

1. Dr. P. H. Daus: *On the solution of the diophantine equation* $x^3 - Dy^3 = \pm 1$. Preliminary report.

The author finds solutions of the equation $X^3 - DY^3 = \pm 1$ from the solutions of the pellian cubic $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$, and shows by means of ternary continued fractions that the first equation cannot have more than one non-trivial solution, and if it has a solution, it is obtainable from the fundamental solution of the pellian cubic.

2. Professor Florian Cajori: *Recent symbolisms for decimal fractions.*

The author describes the struggle for supremacy of the dot and the comma in different countries during the eighteenth and nineteenth centuries, as the decimal separatrix. During the past thirty-five years over half a dozen different decimal notations have appeared in print, viz., 2.5 (in the United States), 2·5 (in Great Britain), 2,5 (the common notation on the European Continent), 2'5, 2‘5, 2,5 (three notations occasionally found in Spanish books), $2,_5$, $2._5$ (two notations found in Scandinavia and Denmark).

3. Professor Florian Cajori: *Pricked letters and ultimate ratios.*

In the PHILOSOPHICAL TRANSACTIONS (London), vol. 3, p. 763, N. Mercator used in 1668 the dot placed over the letter I or over a number, to indicate an infinitesimal difference, similar to the use of the dot in $\dot{x}$ among certain English writers (after Newton in 1693 had used $\dot{x}$ in print to designate a fluxion or velocity) to denote infinitely small quantities. Mercator used in 1668 the Latin forms for "prime" and "ultimate little ratios"; Newton first used in print the phrase "prime and ultimate ratios" in 1687. It is possible that Newton adopted this terminology from Mercator, but his use of the dot in private papers antedates Mercator's use of it in print.

4. Professor B. A. Bernstein: *The complete existential theory of Hurwitz's postulates for abelian groups and fields.*

In a previous paper, the author determined those sets of Hurwitz's postulates for abelian groups and fields that are completely independent. In the present paper he obtains the complete existential theory of Hurwitz's remaining postulate-sets and thus establishes the complete existential theory of all of the sets. The complete paper will appear in this BULLETIN.

5. Professor E. T. Bell: *Periodicities in the theory of partitions.*

Theta functions, not merely theta constants as customary, being used in the analysis of partitions, it is shown that Euler's $P(n)$ is connected with six new denumerants by a remarkable system of periodic relations. Two kinds of periodicity are involved: ordinary (as in analysis); numerical (periodicity of residues with respect to an integer modulus). Either or both can occur in one relation. The theory is closely connected with cyclotomy, Lucas' u_n, v_n, and the algebraic integers of a realm. The paper will appear in the ANNALS OF MATHEMATICS.

6. Professor E. T. Bell: *A revision of the Bernoullian and Eulerian functions.*

Four functions of a single variable are taken as fundamental instead of the usual pairs of Raabe, Schlömilch, Worpitzky, Glaisher and others, and by these the theory is reduced by the symbolic method to a simple and symmetric form.

7. Professor E. T. Bell: *Trigonometry and the numbers B, E, G, R of Bernoulli, Euler, Genocchi, and Lucas.*

All general relations between numbers B, E, G, R of different ranks can be obtained briefly and systematically by Blissard's umbral calculus as an isomorph of trigonometry. A set of 22 such relations involving an arbitrary analytic function is given. The set is complete in that it contains at least one relation of each type. A relation is of type r if it involves precisely r of the symbols B, E, G, R.

8. Professor C. A. Noble: *Retention of a salt solution in a tank of flowing water.*

The author assumes that a salt solution of given strength is continuously admitted into a stream of water which flows uniformly through a system of n tanks; that the contents of each tank is agitated; and that the salt diffuses instantaneously. He deduces the differential equation for the amount of salt remaining in the nth tank at the time t, and solves this equation.

9. Professor A. F. Carpenter: *Generalization of a theorem on flecnode curves of ruled surfaces.*

In this paper the author proves that a necessary and sufficient condition that the planes osculating the flecnode curve and the complex curve of a ruled surface at the four points in which they intersect any ruling form a harmonic pencil is the vanishing of the invariant

$$\frac{p_{21}}{p_{12}}\Delta_1 - \frac{p_{12}}{p_{21}}\Delta_2.$$

This theorem generalizes certain results formerly obtained by the author (TRANSACTIONS OF THIS SOCIETY, vol. 16 (1915), p. 509). The paper further determines the differential equations of the ruled surface generated by the axis of the pencil of osculating planes and shows that if the tangents to one branch of the flecnode curve of the given ruled surface cut out one branch of the flecnode curve of the second ruled surface, the same is true of the tangents to the other branch of the flecnode curve, that a corresponding situation holds for the respective complex curves, and finally that the two ruled surfaces are under these conditions space duals.

10. Professor Florian Cajori: *The history of trigonometric notations.*

The author describes trigonometric notations used by early writers, beginning with Thomas Finck, who in 1583 used contractions *sin.*, *tan.*, *sec.*, *sin.com.*, *tan.com.*, *sec.com.*; he had very few followers in his time. In Great Britain, in 1618 and 1631, severe contractions like *s*, *t*, *s co*, *t co* were introduced by Oughtred, Norwood and others for sine, tangent, cosine, cotangent, but seldom did any two authors or even any one author always adhere to exactly the same set of symbols. About 1760 these severe contractions were abandoned in England. There and on the continent abbreviations of three or four letters (five letters for *cosec*) were usually preferred, but no unanimity exists even today. The origin is traced of notations for the inverse functions (arc sin (Euler), $\sin^{-1}x$ (J. F. W. Herschel), $\frac{1}{\sin}x$ (M. Ohm)), also for hyperbolic functions (sinh (V. Riccati and J. H. Lambert), Sin (C. Gudermann)) and for parabolic functions (J. Booth).

11. Dr. F. R. Morris: *Arrangement of interest tables for Liberty Bonds.*

The usual plan of interest tables for bonds is to place the time until maturity and rate of income in the two margins and the value of the bond in the body of the table. However, it is often desirable to read the rate directly. In the case of Liberty Bonds, compact tables may be constructed such that rate may be read if the market price on a given date is known. With time and rate as arguments the computation is direct; with time and value as arguments the computation is indirect and rather difficult.

12. Professor H. F. Blichfeldt: *Notes on diophantine approximations.* Preliminary report.

The following is a well known theorem: n irrational numbers $\alpha_1, \cdots, \alpha_n$ being given, as well as a positive number ϵ arbitrarily small, then a set of n rational fractions $x_1/y, \cdots, x_n/y$ can be found such that $|\alpha_i - x_i/y| < \epsilon$ and also $< \delta/y^{1+1/n}$ $(i = 1, 2, \cdots, n)$, where δ is a fixed number depending upon n only (cf. TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 234–5; L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 95 ff.). A. Hurwitz has proved (loc. cit.) that $\delta = 1/\sqrt{5}$ for $n = 2$; and H. F. Blichfeldt that $\delta \geqq 1/\sqrt[4]{23}$ for $n = 3$ (proof not yet published). In MATHEMATISCHE ANNALEN, vol. 83 (1921), p. 77 ff., there is a proof by O. Perron that $\delta > (1/n)[0.35/(n+1)]^n$ approximately. The present writer, using Perron's method with refinements based on the principles of the geometry of numbers, obtains the somewhat better result $\delta > (N/\sqrt{n})(2/\sqrt{n})^n$ approximately, where N is a fixed number.

B. A. BERNSTEIN,
Secretary of the Section.

THE EASTER MEETING OF THE SOCIETY

The two hundred twenty-second regular meeting of the American Mathematical Society, being the seventeenth regular Western meeting, and the forty-ninth regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 14 and 15, 1922, in honor of the twenty-fifth anniversary of the Chicago Section. The attendance at these meetings was approximately one hundred fifty, among whom were the following one hundred four members of the Society:

E. S. Allen, F. E. Allen, Baker, Beckwith, Bliss, Blumberg, Bradshaw, Brahana, Brooke, Bussey, C. C. Camp, Carmichael, Chapman, Chittenden, Coble, Crathorne, H. B. Curtis, Curtiss, Dalaker, H. T. Davis, Denton, Dickson, Doll, Dowling, Dresden, Escott, Everett, Eversull, Feldstein-Tartakovsky, Feltges, Fields, Fry, Gibbens, Glenn, Gouwens, Green, W. L. Hart, M. G. Haseman, E. R. Hedrick, Hildebrandt, Hoar, Hodge, Ingraham, Dunham Jackson, Kinney, Lane, Lennes, Logsdon, Lunn, McGaw, M. M. McKelvey, J. V. McKelvey, N. B. MacLean, MacMillan, March, Marshall, T. E. Mason, Meacham, B. I. Miller, E. B. Miller, G. A. Miller, Miser, C. N. Moore, E. H. Moore, E. J. Moulton, F. R. Moulton, A. L. Nelson, Newson, Olson, C. I. Palmer, Pitcher, J. F. Reilly, R. G. D. Richardson, Rider, H. L. Rietz, Risley, Roever, Roman, Schottenfels, Schweitzer, Shaw, W. G. Simon, Skinner, Slaught, Edwin R. Smith, Stecker, Steimley, Stouffer, E. L. Thompson, B. M. Turner, J. S. Turner, Underhill, Wahlin, Walton, Warren Weaver, Wilczynski, Wilder, K. P. Williams, F. E. Wood, Roscoe Woods, J. W. Young, J. W. A. Young, Zehring, Ziwet.

The Council announced the election of the following persons to membership in the Society:

Professor Horace Seely Brown, Hamilton College;
Miss Bess Marie Eversull, University of Cincinnati;
Rev. Francis Joseph Gerst, Loyola College, Baltimore;
Professor Eleanora Harris, Central Missouri State Teachers College;
Professor Albert Anton Heinz, Tsing Hua College;
Miss Anna Margaret Mullikin, Oak Lane, Pa.;
Mr. Percy Charles Herbert Papps, Mutual Benefit Life Insurance Company, Newark;
Professor Harry Munson Showman, University of California, Southern Branch;
Mr. Willis Whited, Consulting Bridge Engineer, Pennsylvania State Highway Department;
Professor Emma Kirtland Whiton, University of Redlands.

Professor A. B. Coble was reelected a member of the Editorial Committee of the TRANSACTIONS, for a term of three years beginning October 1, 1922. The President was requested to appoint a committee to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

On behalf of the committee that had collected a fund in honor of Professor E. H. Moore, Professor Arnold Dresden tendered the fund to the Society. The trust was accepted, under the conditions of the bequest. A complete account is given elsewhere in the present number of this BULLETIN.

The sessions of Friday morning and Saturday were devoted to the reading of the papers listed below. On Friday afternoon Professor A. B. Coble gave the symposium lecture on *Cremona transformations and applications to algebra, geometry, and modular functions*. The program for Saturday morning was made up of three papers delivered by invitation in special celebration of the anniversary of the Chicago Section.

On Friday evening a dinner was held at the Quadrangle Club at which one hundred six persons were present. Toasts were responded to by Professor Coble who spoke for the Chicago Section, Professor Richardson who spoke for the Society, Professor Ziwet and Mrs. H. B. Newson who spoke for the early members of the Chicago Section, and Professor Hildebrandt who spoke on Professor E. H. Moore as mathematician, teacher and man. At the close of Professor Hildebrandt's remarks, Professor Dresden presented to Professor Moore a beautifully illuminated and bound manuscript containing a statement of the establishment of the Eliakim Hastings Moore fund, of which mention has already been made.

At the meetings of the Society, President G. A. Bliss presided, relieved by Vice-President R. D. Carmichael and Professors A. B. Coble, L. E. Dickson, and E. H. Moore.

Titles and abstracts of the papers read at the various sessions follow below. Miss Carlson's paper was read by Professor Jackson. The papers of Dr. Zeldin, Dr. Wiener, Professors Copeland, Glenn, Crum, and R. L. Moore, Dr. Woods, Dr. Speiser, Mr. Wilder, and Professor Hart were read by title. Professor Carr was introduced by Professor MacMillan, Dr. Speiser by Professor Dickson, and Mr. Wilder by Professor R. L. Moore.

1. Professor R. D. Carmichael: *Abstract definitions of the symmetric and alternating groups and certain other permutation groups.*

The most interesting of the theorems given in this paper is the following: If k operations are subject to the sole defining conditions that each is of order three and the product of each pair of them is of order two, then they generate a group which is simply isomorphic with the alternating group of degree $k + 2$. About a score of theorems are given, each of which affords the abstract definition of some infinite class of permutation groups, several of these theorems being devoted to various abstract definitions of the symmetric and alternating groups.

2. Professor D. R. Curtiss: *On the zeros of successive polars of a binary form.*

A theorem due to J. H. Grace states that if a binary form written as $f(z)$ in non-homogeneous notation is apolar to a form $\phi(z)$, then it has a zero within any circle enclosing all the roots of $\phi(z)$. In the present paper it is shown that this is equivalent to the following statement, which has been proved in another way by Dr. J. L. Walsh: Let $z_1, z_2, \cdots, z_n$ lie within a circular region C and satisfy an equation

$$F(z_1, z_2, \cdots, z_n) = 0,$$

where F is symmetric in the z's and is a linear function of each z_i; then the equation $F(z, z, \cdots, z) = 0$ has at least one root in C. A proof is given by mathematical induction, and the theorem is shown to be a special case of one regarding the zeros of successive polars of a form which can be deduced from results due to Laguerre.

3. Professor D. R. Curtiss: *Relations between kindred P and Q functions.*

In his lectures on the hypergeometric function, Klein stated without proof the theorem that every Q function of order k could be expressed linearly with constant coefficients in terms of $k + 1$ kindred P functions. This statement would seem to have been based on a count of constants, and is, in fact, untrue in certain special cases. The present author gives a proof of its correctness, with a method for obtaining such representations, in case the families concerned are irreducible. This method also applies to many reducible cases. As a corollary it follows that in these reducible cases, as well as in the irreducible case, every $k + 2$ Q functions of order k which have the same exponents are linearly dependent. A similar examination is made of the relations between a Q function and a kindred P function and its derivative.

4. Professor C. N. Moore: *On the equivalence of the Cesàro and Hölder means for multiple limits.*

In a paper previously presented to the Society (this BULLETIN, vol. 25 (1919), p. 257) the author has established a theorem with regard to generalized limits in General Analysis which includes as special cases the Knopp-Schnee-Ford theorem with regard to the equivalence of the Cesàro and Hölder means for series, the analogous theorem of Landau for integrals, and other similar theorems dealing with simple limits. In the present paper this theorem is extended to the case of multiple limits, so that it now includes as special cases corresponding theorems relating to multiple series, multiple integrals, and other multiple limits. The main part of the proof consists in the establishment of an identical relation between Cesàro and Hölder means for multiple limits, which, though considerably more complicated for the general case, is analogous to Schur's identity (MATHEMATISCHE ANNALEN, vol. 74 (1913), p. 452) for the case of simple series.

5. Miss Bess M. Eversull: *On convergence factors in triple series and the triple Fourier series.*

In this paper it is shown that if convergence factors satisfying certain conditions are introduced into a triple series summable ($C1$), the resulting series will converge and will approach the value to which the original series is summable, as the convergence factors approach unity. It is also shown that the Fourier development of a function of three variables that is finite and integrable (Lebesgue) will be summable ($C1$) to the value of the function at every interior point of a region throughout which the function is continuous. The foregoing results are used to show that the formal developments that arise in discussing certain problems in the flow of heat really furnish the desired solution of the physical problem.

6. Professor E. B. Stouffer: *Independent sets of coaxial minors of determinants.*

It is known that there cannot be more than $n^2 - n + 1$ independent coaxial minors of a determinant of the nth order. The author determines two such sets of coaxial minors, and obtains the expressions for the elements of the determinant in terms of the minors belonging to each independent set. If one or more of the columns of the original determinant are replaced by corresponding columns from other determinants of the nth order with independent elements, a set of determinants is obtained for which a complete system of independent coaxial minors is determined. The independence of certain sets of sums of coaxial minors of determinants is also proved. This is of importance in invariant theory.

7. Professor P. R. Rider: *On the minimizing of a class of definite integrals.*

Several problems in the calculus of variations lead to the consideration of a definite integral of the form $\int_{x_0}^{x_1} [(1+y'^2)^m/y'']dx$. These are of peculiar interest because the second derivative appears in the integrand, and comparatively few problems of that kind have been completely solved. For example, Euler's historic problem of finding the curve which with its evolute encloses a minimum area gives rise to the particular case $m = 2$ in the above integral. The case $m = 1$ arises in the problem, treated by Rider and Dunkel, of obtaining the curve which with its caustic encloses a minimum area. In finding the curves of minimum mean radius of curvature with respect to the arc and with respect to the abscissa, we are led to the cases $m = 2$ and $m = 3/2$ respectively. In this paper the author obtains the equations of the extremals or minimizing curves for these cases, and proves sufficient conditions for a minimum. The question of the determination of the arbitrary constants that occur in the equations of the extremals is taken up, and it is shown that an extremal can always be passed through two given points and be made to have prescribed slopes at these points. Various cases of variable end conditions are considered. The theory is illustrated by means of the curve of minimum mean radius of curvature with respect to x.

8. Miss Elizabeth Carlson: *On the approximate representation of periodic functions of two variables.*

It has been proved by D. Jackson (TRANSACTIONS OF THIS SOCIETY, vol. 22) that if $f(x)$ is a given function of period 2π, if n is a given positive integer, and if m is a real number greater than 1, then there exists one and only one trigonometric sum $T_{mn}(x)$ of order n, for which the integral of the mth power of the absolute value of the difference between $f(x)$ and $T_{mn}(x)$ reaches a minimum. The corresponding theorems for functions of two variables are proved in this paper. Also, a sufficient condition for the convergence of $T_{mn}(x,y)$ to $f(x,y)$ is found to be that $\lim_{\delta\to 0} \omega(\delta)/\delta^{2/m} = 0$, where $\omega(\delta)$ is the modulus of continuity of $f(x,y)$.

9. Professor G. A. Miller: *Substitution groups whose cycles of the same order contain a given number of letters.*

The total number of letters in all the cycles of order k contained in the group G may be found as follows: Select a set of cycles of order k composed of all the different cyles of this order found in G. Let λ be the number of the complete sets of conjugates under G contained in this set. The total number of letters in all the cycles of order k found in G is then λg, g being the order of G. The triply transitive group of degree 6 and order 120 is the only group which is not symmetric but has the property that the total number of letters in all of its cycles of the same order is exactly equal to the order of the group for every cycle which appears in the group. A necessary and sufficient condition that a transitive group of degree n contain a complete set of distinct conjugate cycles whose total number of letters is less than the order of the group is that the class of this transitive group be less than $n - 1$. There are exactly six transitive groups which are not alternating but have the property that the total number of letters in all the cycles of the same order save one is the order of the group, while for this one the total number of letters is twice the order of the group.

10. Dr. S. D. Zeldin: *Conformal transformations of linear homogeneous difference equations and their invariants.*

In this paper Dr. Zeldin shows that the most general conformal transformation which leaves invariant a linear homogeneous difference equation of the type $p_0(v)u_{v+n} + p_1(v)u_{v+n-1} + \cdots + p_n(v)u_v = 0$, where u and v are the circular coordinates of a point in two dimensional space, is of the form $v = \varphi(\xi)$, $u_v = cu_\xi$, where $\varphi(\xi)$ satisfies one of the two equations $\varphi(\xi + 1) - \varphi(\xi) = \pm 1$, and c is an arbitrary constant. He also determines the functions of the p's which form independent absolute invariants.

11. Dr. Norbert Wiener: *A new form of integral expansion.*

The author develops the formula

$$f(x) = \lim_{n\to\infty} \frac{1}{n!\int_{-1}^{1} e^{1/(x^2-1)}dx} \int_0^1 \frac{d^{n+1}}{du^{n+1}} \{[(u - x)^n \\ + (u + x)^n]\int_{-1}^{u} e^{1/(u^2-1)}du\} f(u)du$$

for even functions $f(x)$ which are uniformly analytic over $(-1, 1)$. He discusses the conditions of validity of this formula, contrasting it with Fourier expansions. He shows that if it is valid for $x = 0$, $f(x)$ is uniquely determined by its values in the neighborhood of $x = 1$.

12. Professor Lennie P. Copeland: *Note on certain semi-invariants of n-lines.*

In this paper the semi-invariantive conditions that one n-line be a pole-polygon of a second n-line are determined. It is proved that there exist $3 + (m - 2)(m - 1)$ necessary and sufficient conditions that two n-lines be mutual pole-polygons. If one n-line is a pencil, the second is also, and they have coincident vertices.

13. Professor O. E. Glenn: *Residues of figurate numbers.*

Systems of residues (modulo p, a prime) may be developed as follows: Select a sequence (r) of integers, infinite one way, satisfying any law, and employ its terms as the initial numbers of appropriate orders of differences in a table of differences. The numbers in this table are of the form

$$N = \sum_j r_j \binom{m}{j}$$

and their residues are arranged, in general, in symmetrically related geometric figures that may be delimited by formulas. Thus we obtain, for instance, all numbers of the type of N which are divisible by p. Among the results of the paper is given a determination of all combinatory numbers

$$\binom{m}{j}$$

which are divisible by p^w and not by p^{w+1}.

14. Professor W. L. Crum: *Inter-variate correlation and the successive measures of dispersion in an ordered statistical series.*

In this paper formulas are developed for calculating the coefficients of correlation between the variates and (1) those next preceding, (2) those preceding by two time-intervals, and so on up to k time-intervals. It is shown that these formulas are expressible in terms of the standard deviations of the series of original items and of their successive finite differences. Several of the correlation coefficients are worked out for particular illustrations from historical statistics. The writer then presents the hypothesis that we may measure dispersion in an ordered series by a succession of numbers, beginning with the standard deviation and including the successive inter-variate correlation coefficients as far as we like. If the correlation coefficients fall off rapidly, the series is highly unstable; if they remain nearly equal to the basic coefficient, the series is relatively stable.

15. Professor W. L. Crum: *Inter-variate partial regression equations in an ordered statistical series.*

This paper seeks to examine the phenomena of fluctuation in an ordered series from a somewhat different point of view from the usual attempt to discover fundamental periodicities. Instead of considering that there is a more or less closely periodic tendency for values of the variable to repeat themselves, the paper studies the hypothesis that there is an average tendency for the items of a series to bear a definite relation to the items next preceding, to those next-but-one preceding, etc. The obvious method for discovering such a relation is by the use of partial regression equations, and general properties of these are developed for the kind of series under consideration. Application of the theoretical results is made to certain historical economic series with a view to testing the validity of the hypothesis and to determining its practical utility in the problem of prediction in time series.

16. Professor R. L. Moore: *Concerning relatively uniform convergence.*

In this paper the author shows that if a sequence of measurable functions converges at every point of an interval I, then it converges relatively uniformly over some subset of I whose measure is identical with that of I.

17. Professor R. L. Moore: *On the cut-points of continuous curves and of other connected point sets in space of two dimensions.*

A proper subset K of a point set M is said to disconnect M in the strong sense provided $M - K$ is not connected. It is said to disconnect M in the weak sense provided not every two points of $M - K$ can be joined by a *closed* and connected subset of $M - K$. The present paper contains, among other results, the following:

(I) No bounded, closed and connected point set M contains a closed and connected subset K which contains an uncountable set of points each of which disconnects M, but not K, in the strong sense.

(II) In order that a closed, connected and bounded point set M shall be a continuous curve which contains no simple closed curve it is necessary and sufficient that every closed and connected subset of M contain uncountably many points each of which disconnects M (in the strong sense).

(III) In order that the continuous curve M shall contain no simple closed curve it is necessary and sufficient that if K denotes the set of all those points $[X]$ of M such that X disconnects M then no subset of K shall disconnect M, even in the weak sense.

18. Professor F. E. Carr: *A solution of a spinning oblate spheroid two body problem.*

The problem deals with the precessional movement of the spheroid and was suggested by the non-rigorous treatment which the earth's precession received from Poisson, Serret and others. The solution for p, the precession, has the form $p = P_0 t + P$, where P_0 is a power series with constant coefficients and P a power series with periodic coefficients, in a parameter denoting the eccentricity of the spheroid. The solution for the motion of the second body is analogous to and a generalization of one of the solutions obtained by W. D. MacMillan, and reduces to that solution for a special case. The solution is not general, but arises from a special set of initial conditions. When applied to the system of the earth and moon and the earth and sun, the solution gives results that agree closely with observed values.

19. Dr. Roscoe Woods: *The elliptic modular functions associated with the elliptic norm curve E^7.*

Professor Bessie I. Miller has discussed (TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 259) the elliptic norm curves E^n for $n = 3$, 4, and 5, for which cases the genus of the associated modular group is zero. In the present paper, which will appear in full in an early number of the TRANSACTIONS OF THIS SOCIETY, the case $n = 7$, for which the genus is 3 and which is fairly typical of the general case, is considered.

20. Dr. Andreas Speiser: *Die Zerlegung von Primzahlen in algebraischen Zahlkörpern.*

In this paper, which will appear in an early number of the TRANSACTIONS OF THIS SOCIETY, the problem of the decomposition of a prime number in an algebraic number domain is shown to be identical with a certain problem in the theory of linear substitutions in a Galois field.

21. Professor G. A. Bliss: *A boundary value problem in the calculus of variations.*

For problems of the calculus of variations with fixed end-points it has been shown by several authors that the Jacobi necessary condition for a minimum is closely associated with a boundary value problem of the linear differential equation theory. In the present paper an analogous situation is studied for a case when the end-points are variable, and the correspondences between the two theories are exhibited.

22. Professor J. F. Reilly: *Certain generalizations of osculatory interpolation.*

The Sprague-King formula for osculatory interpolation makes use of differences to order five, and requires that the curve have second order contact with the partial interpolation curves at the extremities of the interval to which the formula is applicable. The author shows how this formula can be generalized first to make use of differences to order $2h + 1$, where h is a positive integer; and second to require that the curve have contact of order k with the partial interpolation curves. Further, it is indicated how these generalizations apply to formulas employing central differences of four different types: (1) the ordinary central differences of y_h; (2) the central differences relative to the interval between y_h and y_{h+1}; (3) the odd central differences relative to the interval between y_h and y_{h+1}, and the even central differences of y_h, as used by Gauss and Karup; and (4) the even central differences of y_h and y_{h+1}, as used by Everett and Buchanan.

23. Professor Arnold Dresden: *A report on the scientific work of the Chicago Section,* 1897–1922.

This paper appears in the present number of this BULLETIN.

24. Professor J. B. Shaw: *On functional transformations.*

The first part of the paper is concerned with the structure of linear operators of the type representable by integral transformations, the second part with functions of these operators, and the third part with algebras of such operators built upon a given fundamental complex.

Examples are used to introduce the notions such as: (1) $\int_{-1}^{+1}(9x^2y^2 - 3x^2 - 3y^2 + 2xy + 2)()dy$ with fundamental functions 1, x, $(3x^2 - 1)$. (2) $\int_{-1}^{+1}(15x^2y^3 - 9x^2y + 3xy^2 - 5y^3 + 4y - x)()dy$ with fundamental functions 1, x, $(3x^2 - 1)$, $(5y^3 - 3y)$. In case (1) the transformation converts the functions into numerical multiples of themselves; in case (2) it converts each into a multiple of the preceding, and 1 into 0. (3) $\int_0^1[x(1 - y), y(1 - x)]()dy$ with normalized fundamental functions $\sqrt{2} \sin n\pi y/n^2\pi^2$, a denumerable infinity. (4) $\int_0^\infty \frac{1}{2}e^{-|x-y|}()dy$ with normalized fundamental functions $\sqrt{2} \cos (x \cot \alpha - \alpha)(\sqrt{\pi} \sin \alpha)$, a continuous infinity of fundamental functions. (5) $\int_0^x$ and $()dy$ with the fundamental functions $x^{\alpha+n}$ where $0 \leqq \alpha < 1$, and n runs from 1 to ∞, integrally.

For each of these operators an identity transformation exists which gives the so called expansion of the function in terms of the fundamental functions.

In the second part of the paper the real character of the various solutions of integral equations is brought out, such as the Neumann solution, Fredholm's solution, Volterra's solution, and the linear algebra solution. In the third part the use of a given set of fundamental functions belonging to a given transformation in determining other related transformations is shown.

25. Professor E. H. Moore: *On the determinant of an hermitian matrix with quaternionic elements. Definition and elementary properties with applications.*

This paper will appear in an early number of this BULLETIN.

26. Dr. T. C. Fry: *Trigonometric expansions of aperiodic functions.*

At the summer meeting of the American Mathematical Society at Wellesley in September, 1921, a discussion arose regarding methods for determining the amplitudes of sinusoidal components in functions which are not themselves periodic. In the course of this discussion a paper by W. L. Hart (*On trigonometric series*, ANNALS OF MATHEMATICS (2), vol. 18, p. 99) was frequently mentioned and the criticism was made that the restrictions imposed by Hart rather seriously interfered with the usefulness of the results. In the present paper the author attacks this problem from the standpoint of the theory of divergent limits and removes the most serious of these restrictions.

27. Mr. R. S. Hoar: *Mathematical paradoxes involved in the new Bucyrus gasoline shovel.*

The new gasoline shovel just put out by the Bucyrus Company is the first one-engine shovel of any size. Its peculiar construction involves enough apparent mathematical paradoxes to furnish the material for a large part of a course in mechanics. For example, the digging part of the shovel is forced against the bank by the weight of the bank, rather than by the engine. Also, it has a one-part hoist which is apparently a two-part hoist, and the direction of the hoisting force is at an angle of about 45 degrees from its apparent direction. The object of this paper is to place these paradoxes at the disposal of teachers of mechanics, and to inject a bit of mathematical recreation into the programme.

28. Professor E. W. Chittenden: *On permutable quadratic forms in infinitely many variables.*

The first part of the paper contains a determination of the properties of the most general limited quadratic form $K(x)$ in infinitely many variables permutable with a given limited form $H(x)$. The results obtained are applied in the second part of the paper to obtain the solutions of non-linear matricial equations.

29. Professor J. S. Turner: *A fundamental system of invariants of a modular group of transformations.*

In this paper the author considers the group H of all linear homogeneous transformations $x' \equiv ax + by$, $y' \equiv cx + dy$, $ad - bc \equiv 1 \pmod{p^2}$, where a, b, c, d are integers and p is an integral prime, and shows that any invariant $I(x, y)$ of this group is an invariant of the group G: $x' \equiv a_1x + b_1y$, $y' \equiv c_1x + d_1y$, $a_1d_1 - b_1c_1 \equiv 1 \pmod{p}$, where a_1, b_1, c_1, d_1 are integers. It is proved $I(x, y) \equiv R(L, Q) + pF(x, y) \pmod{p^2}$, where L, Q are the known fundamental invariants of the group G, and R, F denote rational and integral functions of their arguments with integral coefficients. By means of the transformation $x' \equiv x + py$, $y' \equiv y \pmod{p^2}$, it is then proved that $R(L, Q) \equiv R_1(L^p, Q^p)$, $F(x, y) \equiv F_1(L, Q) \pmod{p}$. Finally, it is proved that $L^p, Q^p, pL^\alpha Q^\beta$, where $\alpha, \beta = 0, 1, \cdots, p - 1$, α, β, not both zero, form a fundamental system of (independent) invariants of the group H.

30. Professor F. H. Hodge: *Note on a generalization of the strophoid.*

A circle tangent to the Y axis is drawn. This circle plays the same rôle in the determination of the curve in question as that taken by the Y axis in the ordinary construction of the strophoid. This leads to a curve of the sixth order. If the radius of the circle is increased indefinitely, this curve approaches the strophoid as the limiting case.

31. Professor E. P. Lane: *Ruled surfaces of Green-reciprocal congruences.*

The author considers an arbitrary non-developable surface S and associates with this surface an arbitrary Γ'-congruence and the reciprocal Γ-congruence, in the sense of G. M. Green. Corresponding to an arbitrary curve on S there is a ruled surface R' of Γ'; the lines of Γ reciprocal to the generators of R' form the corresponding ruled surface R. The present paper investigates the properties of such corresponding ruled surfaces. The formulas herein developed furnish, in particular, a very brief and elegant method of determining those ruled surfaces R' and R which are developable. The theorem of Green, that the conjugate of a Γ-tangent intersects the corresponding generator of the corresponding developable R in a focal point, is generalized in the form: *The conjugate of a tangent to the curve on S that corresponds to an arbitrary ruled surface R of the Γ-congruence intersects the corresponding generator of R at the point where R is touched by the corresponding tangent plane of S.* A class of curves called *intersector curves* is defined for R' and R. Applications are made to asymptotic ruled surfaces R' and R, which correspond to asymptotic curves on S. These, with the canonical quadric, serve to characterize the pair of covariant reciprocal congruences defined, in Green's notation, by the conditions $\alpha_u = \beta_v = 0$.

32. Professor K. P. Williams: *The Laplace-Poisson mixed equation.*

The author investigates the analytic character of the formal solutions which Borden has obtained for the Laplace-Poisson mixed equation

$$f'(x+1) + p(x)f'(x) + q(x)f(x+1) + r(x)f(x) = 0.$$

The asymptotic form of the solutions is also considered.

33. Mr. H. T. Davis: *A criterion from integral equations relating to the existence of solutions for the one-dimensional boundary value problem.*

In this paper the known existence of solutions of a linear homogeneous differential equation with a parameter of the Sturmian type for one set of boundary conditions is used to prove the existence of solutions which satisfy a second set of boundary conditions. If $G_1(x, t)$ is the Green's function for the first system, and $G_2(x, t)$ the Green's function for the second system, then $\int_a^b [G_1(x, x) - G_2(x, x)]dx \neq 0$ is shown to be a sufficient condition that the values of the parameter for which solutions exist for the two problems shall either coincide or alternate with one another.

34. Mr. H. T. Davis: *A general criterion relating to the existence of solutions for the one-dimensional boundary value problem.*

Suppose F_1 is the determinant whose zeros are characteristic values of one boundary value problem, and F_2 the determinant belonging to a second boundary value problem. The product $F = F_1 \cdot F_2$ is used to furnish a criterion for the existence of solutions for one system, supposing the existence of solutions has already been proved for the other. An application is then made to the general self-adjoint system of second order.

35. Mr. R. L. Wilder: *A continuous curve in the rôle of a space.*

The author shows that if the set of points constituted by a continuous curve is regarded as a space S, then if certain terms are suitably defined, many theorems which are true in ordinary space hold true for the space S.

In particular, it is shown that Theorems 1–16 of R. L. Moore's paper *On the foundations of plane analysis situs* (TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164) hold true. In addition, the following theorems are proved: (1) If R is a domain with respect to S, and B, the boundary of R with respect to S, is connected im kleinen, then any point P in B can be joined to any point in R by a simple continuous arc lying wholly in R except for the point P. (2) In order that a closed and bounded point set shall be connected im kleinen, it is necessary and sufficient that it be the sum of a finite (or vacuous) set of continuous curves, together with a finite (or vacuous) set of isolated points.

36. Professor N. J. Lennes: *Continuous transformations in analysis situs.*

For a given Jordan curve j and a fixed point P within it, this paper describes the construction of a set $[j]$ of Jordan curves such that (1) every curve lies within j and contains P as an interior point; (2) through every point within j, except P, there is one and only one curve of the set $[j]$. If any curve j_1 of the set $[j]$ is removed, the remainder of the set consists of two sets of curves $[j]_1$ and $[j]_2$ such that (*a*) no point in either set is a limit point of points in the other; (*b*) every point of j_1 is a limit point of points in $[j]_1$ and also of points in $[j]_2$. The set of curves $[j]$ is said to constitute a *continuous transformation* of any one of its curves into any other or into the point P.

Such a set can be constructed transforming any plane curve into any other provided the curves have no point in common and one of the curves does not have a point interior to and another point exterior to the other curve.

The problem is proposed to construct a continuous transformation of a *doubly closed continuous curve not lying in a plane* into a plane Jordan curve. This problem, connected with the theory of "knots," is not completely solved.

37. Professor N. J. Lennes: *On the foundations of the theory of sets.*

In the MATHEMATISCHE ANNALEN, vol. 65, E. Zermelo gives a set of axioms for sets (Mengen), one of his purposes being to formulate a theory which should not involve the contradiction that has been shown to follow from Cantor's classical definition of "Menge." The purposes of the present paper are: (1) to show that though it follows from Zermelo's axioms that there are sets having cardinal numbers such as that of the continuum and higher, it cannot be shown that an arbitrary collection of objects more than finite in number is a set; (2) to modify Zermelo's theory so as to identify as a set any collection of objects having the same cardinal number as that of some Zermelo set; (3) to study the independence of Zermelo's axioms. It is proved that except for the "axiom of choice" (Axiom der Auswahl) Zermelo's axioms are independent. Similar independence proofs are given for the axioms with which it is proposed to replace Zermelo's set.

38. Professor L. E. Dickson: *An error in the theory of differential equations by Lie's method.*

In Lie-Scheffer's *Differentialgleichungen*, pp. 412–433, it is proved that every differential equation of the second order in two variables which is invariant under two infinitesimal transformations generating a two-parameter group can be integrated by quadratures, except in the last of four cases, when it is also necessary to integrate an auxiliary differential equation of the first order. But the last case is not exceptional, and also requires only quadratures. The oversight was due (p. 424) to not making full use of the hypothesis $(U_1U_2) \equiv U_1f$ of that case. In fact, the long Chapter 18 can be compressed into two pages by noting that under the former hypothesis or the remaining one $(U_1U_2) \equiv 0$, the infinitesimal transformation U_2f leaves invariant the partial differential equation $U_1f = 0$, whence a solution ϕ of the latter may be found by quadratures. When ϕ and y are taken as new variables, U_1f becomes $F\partial f/\partial y$, which is reduced to $\partial f/\partial y_1$ by choice of a function y_1 of ϕ and y found by an obvious quadrature. It is now a simple matter to obtain the four types of canonical forms of U_1, U_2 by quadratures only.

39. Professor L. E. Dickson: *Present status of the history of the theory of numbers.*

The purpose of this preliminary communication is to announce (1) that the third volume of the writer's *History of the Theory of Numbers*, entitled *Arithmetical theories of quadratic and higher forms,* is being put in type for the Carnegie Institution of Washington, and (2) that the publication will soon be undertaken by the National Research Council of a report prepared by Professors Dickson, H. H. Mitchell, Vandiver and Wahlin on algebraic numbers and related topics, chiefly from 1894 to date, with the primary aim to supplement the report by Hilbert in the yearbook of the German Mathematical Society for 1894.

40. Professor W. L. Hart: *The determination of a seasonal variation.*

Let $y = f(t)$ represent a statistical table, in which $t = 0, 1, \cdots, 12k$ months, where $t = 0$ is January, first year, $t = 15$ is March, second year, etc. The method of monthly means for determining the seasonal variation present in the function $f(t)$ consists of forming a January entry equal to the arithmetic mean of all the given January values, $\cdots$, a December entry equal to the mean of the December values. The author states certain useful properties of this method. He also gives an obvious justification of its use in the case of economic data, affected by the business cycle, by virtue of a certain well known property of the trigonometric functions. Attention is called to the relatively complicated character of the method for determining seasonal variation which has been used by Professor W. M. Persons, and an example is considered which illustrates the possible inaccuracy of his method.

41. Dr. V. D. Gokhale: *Concerning compact Kürschák fields.*

In the paper *Ueber Limesbildung und die allgemeine Körpertheorie* (JOURNAL FÜR MATHEMATIK, vol. 142), Kürschák sets up abstract fields with a modulus ("bewertete Körper"). This modulus plays, in the general field, essentially the same rôle as the absolute value in the fields of classical analysis, viz., real number system, complex number system, etc. Kürschák proves that every abstract field of this type has the (smallest) algebraically closed and perfect extension.

In the present paper the author sets up the notion *compactness.* This notion is analogous to M. Fréchét's compactness and to the J-compactness in E. H. Moore's General Analysis. It is a generalization of the following property in the point set theory: Every infinite set of points in a bounded domain has at least one condensation point. He then studies the properties of algebraically closed and compact fields and compactness under the adjunction of algebraic elements. Using Ostrowski's results, he proves the theorem that the smallest algebraically closed extension of a compact field is compact if and only if it can be obtained by adjoining a single algebraic element. The last part of the paper develops a complete existential theory of the four properties: (1) having a characteristic other than zero, (2) algebraic closure, (3) perfection and (4) compactness. Out of the $2^4 = 16$ possibilities 11 are shown to be existent and the remaining 5 non-existent.

42. Professor W. H. Roever: *A second mechanism for illustrating lines of force.*

Several years ago the author designed a mechanism for illustrating certain systems of lines of force and stream lines (see ZEITSCHRIFT FÜR MATHEMATIK UND PHYSIK, vol. 62 (1914), and also BULLETIN OF THE MOUNT WEATHER OBSERVATORY, vol. 6 (1914), Part 5). In the mechanism exhibited at this meeting, one of the two wheels of the former mechanism is replaced by a striped curtain which moves with uniform linear velocity just behind a uniformly rotating spoked wheel. For sufficiently rapid motions of the curtain and wheel the paths of the points of intersection of the spokes of the wheel with the stripes of the curtain become distinctly visible. These curves were shown to be identical with lines of force of the field resulting from the introduction of a source (or a sink) in an originally constant field of force.

ARNOLD DRESDEN, R. G. D. RICHARDSON,
Secretary of the Chicago Section. *Secretary of the Society.*

A REPORT ON THE SCIENTIFIC WORK OF THE CHICAGO SECTION, 1897–1922 *

From December, 1896, forward, meetings were held in or near Chicago twice a year regularly, in the spring and during the Christmas holidays.

The number of papers presented at these meetings of the Section is 1,102. If we classify these papers on the basis of the encyclopedic classification, the result is as follows:

Arithmetic and Algebra	335 or	30	per cent.
Analysis	380 "	35	"
Geometry	248 "	22	"
Mechanics (incl. Math. Physics)	93 "	8	"
History, Pedagogy, Philosophy	46 "	4	"

It is found that the 335 papers on arithmetic and algebra were presented by 84 authors, giving an average of 4 papers per author; the 380 papers in analysis, by 135 authors, which is slightly less than 3 papers per author; the 248 papers in geometry, by 118 authors; in mechanics 93 papers, by 33 authors; in history, etc., 46 papers, by 18 authors. While thus the average number of papers presented by each author in each separate field is between 3 and 4, this average is very misleading. For in the first place, the same author occurs frequently in more than one group, the total number of authors being 278, giving a general average of 4 papers per author. But also this figure gives a very wrong impression of the relative distribution of the papers. For out of the 278 authors, about one half, exactly 133, presented one paper each, i.e., about 10 per cent of the papers. More striking still is the fact that there is a group of 10 authors, constituting less than 4 per cent of the total number, who have contributed 329 papers, which is about 30 per cent of the total; and finally, there is one author who has to his credit 70 of the titles, or over 6 per cent of the total. In view of such extremes, averages mean very little. If one groups the authors according to the number of papers they have presented, the

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* Parts of a report presented before the Society on April 15, 1922.

number in each group decreases very rapidly as the number of papers per author increases. The result is as follows:

133	authors having presented	1	paper each	133
43	“ “ “	2	“ “	86
24	“ “ “	3	“ “	72
12	“ “ “	4	“ “	48
11	“ “ “	5	“ “	55
14	“ “ “	6	“ “	84
5	“ “ “	7	“ “	35
3	“ “ “	8	“ “	24
9	“ “ “	9	“ “	81
1	“ “ “	10	“ “	10
3	“ “ “	11	“ “	33
5	“ “ “	12	“ “	60
1	“ “ “	13	“ “	13
1	“ “ “	14	“ “	14
1	“ “ “	15	“ “	15
2	“ “ “	16	“ “	32
1	“ “ “	19	“ “	19
1	“ “ “	20	“ “	20
1	“ “ “	21	“ “	21
1	“ “ “	24	“ “	24
1	“ “ “	27	“ “	27
1	“ “ “	32	“ “	32
1	“ “ “	35	“ “	35
1	“ “ “	39	“ “	39
1	“ “ “	42	“ “	42
1	“ “ “	70	“ “	70

The 10 authors coming last in this group are:

W. D. MacMillan
E. H. Moore
A. R. Schweitzer
L. E. Dickson
G. A. Bliss
E. J. Wilczynski
J. B. Shaw
F. R. Moulton
R. D. Carmichael
G. A. Miller

If now we inquire as to the publication of these papers, we find that in arithmetic and algebra there have been published 213 out of 335 (64 per cent), in analysis 217 out of 380 (57 per cent), in geometry 147 out of 248 (59 per cent), in mechanics 49 out of 93 (53 per cent) and in history, etc., 23 out of 46 (50 per cent)—a total of 649 out of 1,102 (59 per cent).

I shall not attempt to evaluate the relative importance for the work of the Section of published versus unpublished papers. Doubtless there are many papers that have fulfilled their total possible usefulness to author and public when once presented. I do not believe therefore that 100 per cent publication is an end to be too diligently pursued. The meetings of the Section would lose a good share of their value if there were no room for airing views, proposing questions, and making suggestions which might not be suitable to appear in print. On the other hand, a small ratio of published to presented papers would not be a sign of the most wholesome development. The figures presented above give but a rough idea of the general character of the published material because they include a variety of journals of varying grade and standard. For that reason it seemed worth while to single out for separate mention the group of papers published in the TRANSACTIONS. This group consists of 166 papers, which is about one fourth of the published papers, or 15 per cent of the total number of papers presented. It is interesting to note that not a single one of the 21 completed volumes of the TRANSACTIONS is without several Chicago Section papers, the number averaging 8 per volume and running as high as 12 and 13 in some volumes.

I have thus dwelt somewhat at length upon the papers offered for presentation at the meetings of the Chicago Section, because I feel that this aspect of our meetings is perhaps most widely useful. For a mathematician to know that he can find a group of more or less informed colleagues who will at least listen to what he has to say about his own work, and perhaps comment on it with a measure of understanding of what he is working for, encourage him if he deserves it, discourage him if it be otherwise, must be for him a stimulus of which each one of us can best appreciate the value from his own experience. This value, we will agree, I think, depends to some extent upon the quality and the quantity of our auditors. Since of the former of these characters it is difficult for me to speak, I shall make a brief statement about the latter. The attendance of our meetings is very variable. During the first ten years the number of members present never exceeded 30. Since then it has never fallen below 30, jumping up to 84 in December, 1908, when the A. A. A. S. met in Chicago, to fall back to 33 at the following meeting in April, 1909. Since then the attendance at the meetings has grown fairly steadily to an average of about 50. A year ago the number went up to 83, and if our expectations are not disappointed, all previous records will be surpassed on the present occasion. (The attendance at the meeting of April 14 and 15 included 104 members.) But this growth in numbers is but an outward sign of an inner development. And this inner development was the result of careful and thoughtful nurture on the part of those who carried the responsibility.

At the meeting of April, 1897, the Chicago Section was formally organized with E. H. Moore as chairman, T. F. Holgate as secretary, and Professors Ziwet and Hathaway as members of the program committee. In December, 1905, Professor H. E. Slaught was elected secretary of the Section, and he continued to serve in that office for ten years, until December, 1915. No one can appreciate the devotion which he gave to his work as secretary more than his successor does, unless it be those who were associated with him on the program committee. It was during the second decade of its existence that the Section took its big strides forward and all those who, like myself, began to attend the meetings during that time know to how large an extent it was his work that led it forward.

The Section entered upon a period of vigorous development in 1905. A committee was appointed to consider means of improving the meetings of the Section. As a result of the work of this committee, there were held in December, 1907, joint meetings with Sections A and D of the A. A. A. S., including a symposium on the teaching of mathematics to engineers. The following year the Section appointed a committee to study the possibility of improving the character of the mathematical appointments in our colleges and universities, resulting in a report adopted by the Section and a tentative plan proposed by Professor Wilczynski and published in the BULLETIN. Joint meetings with the A. A. A. S. and with engineering organizations were held in Minneapolis in 1910, in Cleveland in 1912, in Columbus, Ohio, in 1915, in St. Louis in 1919, and in Toronto in 1921. In April, 1911, the Society met in Chicago to hear Professor Bôcher's presidential address. In 1913 the Council of the Society took official notice of the growing importance of the meetings of the Chicago Section by resolving that "the meetings of the Chicago group are henceforth to be designated as meetings of the Society." From 1911 on the retiring chairman delivered an address. When voting in 1913 to make the chairman's address a permanent feature, the Section expressed the opinion "that there are too few papers giving a general survey of any field of mathematics." For a short period, time was set aside on each program for "informal notes and queries" in the hope that in that way more general participation in the discussion might be secured. These different efforts to give the meetings a broader scope led, in 1915, to a proposal by Professor Van Vleck that a symposium be arranged on some topic of wide interest. But it was not until 1917 that the proposal was carried into effect; at that time Professors Bliss and Hildebrandt gave lectures on the *Lebesgue Integral.* From that time on a symposium address has been given at the April meeting of each year.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

ELIAKIM HASTINGS MOORE FUND

BY ARNOLD DRESDEN

A group of former students of Professor E. H. Moore, wishing to use the opportunity afforded by the twenty-fifth anniversary meeting of the Chicago Section to present to Professor Moore a testimonial of their respect and appreciation, brought together a fund to be used for furthering mathematical interest in this country. This fund was contributed to by one hundred and seventy-four persons, former graduate students of mathematics at the University of Chicago, or members of the American Mathematical Society definitely identified with its Chicago Section. On the evening of April 14, 1922, a beautifully bound and illuminated manuscript containing a statement of the establishment of the Eliakim Hastings Moore Fund was presented to Professor Moore. The names of the contributors to the fund are preceded by the following statement:

"Conscious of the great influence which you have exercised upon the development of mathematical science throughout this country, particularly in the Middle West, during the last twenty-five years,

Admiring the outstanding qualities of your researches in various fields of mathematics,

Grateful for the inspiration and the encouragement which you have given to those who have come to the University of Chicago to study mathematics,

Recognizing the large contribution which you have made to the creation and the growth of the Chicago Section of the American Mathematical Society,

Deeply appreciative of the friendship which, during many years, you have shown toward those who have had the good fortune to know you,

The undersigned members of the American Mathematical Society, formerly students of mathematics at the University of Chicago, or members of long standing in the Chicago Section, have wished to use the opportunity afforded by the twenty-fifth anniversary meeting of the Chicago Section to present to you a testimonial, which is intended to link your name in the years to come with the development of mathematics in this country.

To this end they have contributed to a fund which is to be offered for trusteeship to the American Mathematical Society upon the following conditions:

1. The fund is to be known as the Eliakim Hastings Moore Fund.

2. The interest on the fund is to be used at the discretion of the Council of the Society, and upon the recommendation of a committee appointed from time to time for this purpose, in furtherance of such mathematical interests as

(*a*) The publication of important mathematical books and memoirs;

(*b*) The award of prizes for important contributions to mathematics;

it being further recommended that during the next ten years preference be given to the former, and that publication of Professor E. H. Moore's researches in General Analysis or other fields shall have precedence over all other claims.

3. The fund is to be kept intact by the American Mathematical Society, except in so far as it is used to aid in the publication of Professor Moore's researches. For this special purpose a part of the principal, not exceeding one third, may be used provided the interest on the remainder be allowed to accumulate until the fund has been restored to its original value."

At its meeting on April 15, the Council of the Society accepted the trusteeship of this fund as recorded elsewhere in the BULLETIN.

It is hoped that the Eliakim Hastings Moore Fund, which has now reached approximately $2,000, may be the nucleus of a much larger fund to be ultimately established and to be at the disposal of the American Mathematical Society for aid in the publication of important mathematical work. The fund will remain open for contributions at any time, and it is hoped that those who are interested and who had no opportunity to join in the testimonial at the time of its foundation may do so later. Contributions may be sent to the Secretary of the Society, Professor R. G. D. Richardson, Brown University, Providence, Rhode Island.

THE UNIVERSITY OF WISCONSIN.

NOTE ON THE DIVISION OF A PLANE BY A POINT SET*

BY E. W. CHITTENDEN

A plane set of points K is said to divide a plane S if the set $S - K$ is composed of two mutually exclusive domains S_1, S_2, of which K is a common boundary, where by domain is meant a connected open set. The condition that K be a simple closed curve or an open curve has been stated by J. R. Kline† in terms of the concept "connected im kleinen." In proving that the set K is a connected set, Kline employs the condition "connected im kleinen."

If we assume that K is bounded and that we have at our disposal the parallel and perpendicular straight lines of a number plane, then the connectedness of K is established by other writers, for example Hausdorff, *Grundzüge der Mengenlehre,* page 346, Theorem XII. Hausdorff calls attention, in a footnote to page 342, to the difficulty of extending his argument to the case of unbounded sets.

It seems in view of the importance of the theory of open curves as indicated by R. L. Moore‡ and of the importance in general of the fundamental theorems of plane analysis situs that it is of interest to show that the set K is connected, whether bounded or not, without the use of the restriction employed by Kline or of the properties of straight lines and rectangles. The present note is concerned, therefore, with the proof of the following theorem.

THEOREM. *Let K be a plane point set, S, the set of all points of the plane, and denote by S_1, S_2 two mutually exclusive domains such that*

$$S - K = S_1 + S_2.$$

Then if every point of K is a limit point of both S_1 and S_2, the set K is closed and connected.

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* Presented to the Society Nov. 26, 1921.

† *Concerning approachability of simple closed and open curves.* TRANSACTIONS OF THIS SOCIETY, vol. 21 (1920), pp. 451–458.

‡ R. L. Moore, *On the foundations of plane analysis situs.* TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164. This paper will be referred to as "Foundations."

Proof. The set K is closed. For any limit point of K is evidently a common limit point of S_1 and S_2. But no limit point of S_1 can belong to the domain S_2, and likewise no limit point of S_2 can belong to S_1. Such a point must belong to K.

Assume that the set K is not connected. Then there exist two mutually exclusive closed subsets K_1 and K_2 of K such that $K = K_1 + K_2$. Let P_i denote a point of K_i $(i = 1, 2)$. We may enclose P_i in a region R_i which contains no point of K_{i+1}.* Let J_i be a simple closed curve lying in R_i and containing P_i as an interior point. Let P_{ij} be a point of S_j lying within J_i. Since S_j is a domain, there is an arc

$$P_{1j}\,X\,P_{2j}$$

lying entirely in S_j. Let A_{1j} be the last point which this arc has in common with J_1 and let A_{2j} be the first point which the arc has in common with J_2 after A_{1j} on $P_{1j}\,X\,P_{2j}$.

Since the point A_{ij} lies on the boundary of J_i, it may be connected with P_i by an arc $P_i\,X\,A_{ij}$ which, except for the end-point A_{ij}, lies entirely within J_i. Furthermore the arcs $P_i\,X\,A_{i1}$ and $P_i\,X\,A_{i2}$ may be constructed so that they have no common point besides P_i.

From the arcs so defined we construct a simple closed curve J,

$$J:\ P_1A_{11}A_{21}P_2A_{22}A_{12}P_1.$$

Let H_1 denote the set of all points of K_1 which are interior to J but not interior to J_1. The set H_1 is closed.

CASE I. No point of H_1 lies on J_1. Then the points of J_1 which lie in J lie in $S_1 + S_2$. There exist points of J_1 within J, since by Theorem 40 of the *Foundations* it is possible to join P_1 and P_2 by an arc $P_1\,X\,P_2$ such that $\underline{P_1\,X\,P_2}$† lies entirely within J. This arc must meet J_1 in at least one point. It follows readily that there is an arc $A_{11}\,X\,A_{12}$ of J_1 which lies in J and therefore in $S_1 + S_2$. This is contrary to Lemma A of the paper of J. R. Kline.‡

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* The subscripts are to be reduced modulo 2.

† The symbol $\underline{A\,X\,B}$ denotes the set $A\,X\,B - A - B$, that is, the set of all points of the arc $A\,X\,B$ except its end-points.

‡ Loc. cit., p. 452. Every arc joining a point of S_1 to a point of S_2 contains a point of K.

CASE II. The curve J_1 contains a point of H_1. From the Heine-Borel property we may assume a finite set of regions

$$(1) \qquad \bar{R}_1, \bar{R}_2, \cdots, \bar{R}_m,$$

covering H_1. We may without loss of generality assume that the boundary of each region $\bar{R}_k$ $(k = 1, 2, \cdots, m)$ is a simple closed curve $\bar{J}_k$. We will also assume that no point of J, J_2, or K_2 lies in or on the boundary of any of the regions $\bar{R}_k$. By hypothesis some of the regions $\bar{R}_k$ have points in common with the interior of J_1. Let

$$(2) \qquad \bar{R}_1, \bar{R}_2, \cdots, \bar{R}_p, \quad (p \leqq m),$$

be a subset of the regions (1) such that $\bar{R}_1 + \cdots + \bar{R}_p$ forms, with the interior of J_1, a connected set. Then the curves $J_1, \bar{J}_1, \bar{J}_2, \cdots, \bar{J}_p$ form a finite family G of closed curves whose interiors form a connected set. By theorem 42 of the *Foundations* there is a simple closed curve $\bar{J}$ which satisfies these conditions: every point of $\bar{J}$ belongs to some curve of G; the interior of $\bar{J}$ contains the interiors of all the curves of the set G.

The curve $\bar{J}$ meets J in the points A_{1j} and in no other points. We may show as in Case I that there is a point X on $\bar{J}$ which is interior to J, and that the arc $\underline{A_{11}\,X\,A_{12}}$ of the curve $\bar{J}$ lies within J.

We will show that no point of $A_{11}\,X\,A_{12}$ is in K. By construction no point of K_2 lies on any curve of the set G. Suppose Q, a point of K_1, lies on the arc $A_{11}\,X\,A_{12}$. Then Q must lie on one of the curves of the set G. If Q is on J_1, it must belong to H_1 since Q is interior to J. Consequently Q is in H_1 in any case and must lie in some region $\bar{R}_k$ $(k \leqq m)$ of the regions (1). Suppose that Q lies on $\bar{J}_q$ $(q \leqq p)$. Then R_k must contain an interior point of $\bar{J}_q$. But the interior of $\bar{J}_q$ is connected with R_1. Consequently R_k is connected with R_1 and is interior to $\bar{J}$. It follows that Q is an interior point of $\bar{J}$, contrary to hypothesis.

Since no point of K lies on $A_{11}\,X\,A_{12}$, we have obtained an arc connecting a point of S_1 with a point of S_2 which contains no point of K. This again contradicts Lemma A.

The proof that the set K is connected is completed.

THE UNIVERSITY OF IOWA

NOTE ON STEADY FLUID MOTION

BY S. D. ZELDIN

In a previous paper* I have shown how to find special invariant configurations of the projective and linearoid groups investigated by Wilczynski† in connection with steady fluid motion. It is the purpose of this note to show how the group whose general infinitesimal transformation is

$$Kf = u(x)\frac{\partial f}{\partial x} + v(x, y)\frac{\partial f}{\partial y} + w(x, y, z)\frac{\partial f}{\partial z}‡$$

should be simplified in order to represent the steady motion of a fluid under the influence of forces possessing a potential.

If the external forces have a potential, then, as is known, the functions u, v, w must be such that the expression

$$Kudx + Kvdy + Kwdz$$

is a complete differential, or, what amounts to the same thing,

$$\begin{aligned} &\frac{\partial Ku}{\partial z} - \frac{\partial Kw}{\partial y} = 0, \\ (1) \qquad &\frac{\partial Kw}{\partial x} - \frac{\partial Ku}{\partial z} = 0, \\ &\frac{\partial Ku}{\partial y} - \frac{\partial Kv}{\partial x} = 0. \end{aligned}$$

Performing the operations indicated by equations (1) we get:

$$(a) \qquad \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} = 0,$$

$$\begin{aligned} (b) \quad \frac{\partial u}{\partial x} \cdot \frac{\partial w}{\partial x} + u\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial x \partial y} + w\frac{\partial^2 w}{\partial x \partial z} & \\ + \frac{\partial v}{\partial x} \cdot \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial z} &= 0, \end{aligned}$$

$$(c) \quad u\frac{\partial^2 w}{\partial x \partial y} + v\frac{\partial^2 w}{\partial y^2} + w\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial v}{\partial y} \cdot \frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} \cdot \frac{\partial w}{\partial z} = 0,$$

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* JOURNAL OF MATHEMATICS AND PHYSICS (Mass. Inst. of Tech.), vol. 1 (1921), p. 54.

† TRANSACTIONS OF THIS SOCIETY, vol. 1 (1900), pp. 339–352.

‡ This class of groups has been investigated by Sophus Lie in connection with two-point invariants. See Lie-Engel, *Theorie der Transformationsgruppen*, vol. 3, Abtheilung 5.

which can be written in the form

$$(a)\qquad \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right)=0,$$

$$(b)\qquad \frac{\partial}{\partial x}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=0,$$

$$(c)\qquad \frac{\partial}{\partial y}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=0.$$

From (a) follows at once that Kv *can be a function of* y *only.* From (b) we see that Kw must be a function of y and z alone, and, since by virtue of (c) Kw must be a function of x and z alone, it follows that Kw *can be a function of* z *only.*

If the fluid is incompressible, we have the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

and therefore w is a *linear function* of z.

In the case of irrotational motion, since

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0,\qquad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0,\qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0,$$

we must have, in the above infinitesimal transformation Kf, u *a function of* x *alone,* v *a function of* y *alone,* and w *a function of* z *alone.* There exists then a velocity potential, say F, where

$$u=\frac{\partial F}{\partial x},\qquad v=\frac{\partial F}{\partial y},\qquad w=\frac{\partial F}{\partial z},$$

and the orthogonal trajectories of the family of surfaces

$$F=\int u(x)dx+\int v(y)dy+\int w(z)dz=\text{constant}$$

represent the stream lines. These stream lines are the intersections of the two families of cylinders obtained by solving the equations

$$\frac{dx}{u(x)}=\frac{dy}{v(y)}=\frac{dz}{w(z)}\,.$$

The separately invariant points will be found by solving the equations

$$u(x)=0,\qquad v(y)=0,\qquad w(z)=0.$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

TWO BOOKS ON ANALYSIS

Lezioni di Analisi Infinitesimale. By Giulio Vivanti. 2d edition. Torino, S. Lattes and Co., 1920. vii + 693 pages.

Lezioni di Calcolo Infinitesimale. By Ernesto Pascal. Milano, Ulrico Hoepli. Part I, 4th edition, 1919, viii + 330 pages. Part II, 4th edition, 1918, xii + 313 pages. Part III, 2d edition, 1918, xi + 325 pages.

Permitting oneself a variation on a well known theme, one could perhaps say to a nation: "Show me your fundamental course in analysis and I will tell you who you are mathematically." For doubtless, a valuable estimate of a nation's state of development in mathematics can be obtained by considering the form in which the fundamentals of the infinitesimal calculus are presented to the students of that nation.

If the two works under review may be taken as representative Italian courses, one must form a high estimate of the esteem in which mathematics is held in that country. In scope, point of view and method of approach, they are broad and scholarly. They cover practically the same ground except for Pascal's third volume which is devoted entirely to the calculus of variations and to the calculus of finite differences. They are not encyclopedic like the French cours d'analyse; neither are they written on the superficial plan of so many of our American college texts. From them a student can learn enough to be well prepared for special study in analysis, as well as for work in the applied sciences—he will at least have acquired that which he is to apply. Both intended for technical students, the authors do not hesitate to include the elements of the theory of point sets and other topics usually regarded in our classes as material unfit for the training of "practical men." In his preface, written May 1917, Pascal says "It is certain that through the profound changes which the critical spirit has made in the foundations of the calculus, even a course intended for those for whom mathematics is a means rather than an aim, cannot but use the new results which have been reached . . . it would therefore exhibit a shortsighted view and little esteem for the ability of the future engineer, to believe that it would be sufficient for them, at least if they can, to learn to operate the calculus in about the way in which a workman knows how to operate a machine made by others, and of which he does not know the inner connections."

Is not this a point of view worth the consideration of our teachers of engineering students? Not merely for the mathematical specialist, but for the person concerned with the applied sciences, a fundamental theoretical course, not merely a working course, is requisite if these applications are to be more than mere mechanical adaptations of the thoughts of others.

The two books differ in their method of treatment, inasmuch as Vivanti is in favor of and Pascal opposed to the fusion of differential and integral calculus. The *Lezioni* of Vivanti consists of the following six parts: I. Analytical introduction (90 pp.), II. Derivatives and integrals of functions of one variable (150 pp.), III. Derivatives and integrals of functions of several variables (84 pp.), IV. Geometrical applications (183 pp.), V. Differential equations (138 pp.), VI. Calculus of variations (20 pp.).

In the analytical introduction are assembled those notions from the theory of functions of a real variable which form the irreducible minimum for a sound development of the calculus: real numbers, sequences and series, greatest lower and least upper bounds, continuity, uniform convergence, etc. This part reveals what an excellent expositor the author is. The reviewer noticed with pleasure several neat proofs, such as the one on page 39, in which it is shown that $\lim_{x \to c} f(x) = a$ implies that $\lim_{x \to c} b^{f(x)} = b^a$. In the arrangement of the material, in the working out of details, in the style and the form, a masterly hand shows itself. This section provides for the reader an enjoyment which I would only spoil by dwelling on the details.

With such preparation a thorough treatment of the calculus becomes possible. A curious slip occurs on page 140, where from the uniform convergence of a series of functions $f_i(x)$ on an interval (ab), the conclusion appears to be drawn that a series of functional values $f_i(x_i)$ must converge for arbitrary choice of the x_i. It is surprising to find in this book the notation $\partial f(\alpha, \beta)/\partial \alpha$ where $[\partial f(x, y)/\partial x]_{\alpha, \beta}$ is meant, a notation no longer used in our better texts and, in my judgment, very undesirable. Space is lacking for detailed mention of the many instances of elegance in treatment found throughout these parts of the book.

It seems curious that with such free fusion of the two divisions of the calculus, there should be such a sharp separation between the calculus and its geometric applications, to which the largest single part of the book is devoted. In the first 20 pages of this section we find an exposition of the elements of vector analysis which follows the methods of Burali-Forti and Marcolongo. Throughout the rest of the section these methods, as well as the ordinary methods, are used in the applications to differential geometry, which in many instances are improved and simplified in this way. The incorporation of the vector treatments, assembled in an appendix in the first edition of the *Lezioni*, as an organic part of the book in this new edition, bears witness to the vogue which the work of Burali-Forti and Marcolongo has given these methods.

In Part V, particular mention should be made of the simple and straightforward discussion of the singular integral and its geometrical interpretation, as well as of the treatment of the linear homogeneous equation with constant coefficients by the use of simple relations holding between the properties of the linear differential operator and the associated algebraic functions. A brief section devoted to the calculus of variations treats a few of the classical problems and merely mentions some of the others. Numerous examples are worked out in the text. For further exercises the reader should consult the companion volume, the author's *Esercizi*.* The misprints are very few and do not seriously mar the typographical excellence of this book.

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* See the review of the book by R. C. Archibald in this BULLETIN, vol. 20 (1914), p. 482.

In the three volumes of Pascal's *Lezioni* on the other hand, misprints are quite numerous, particularly in volume 3, where some fifty were noticed on slightly over three hundred duodecimo pages. Enough has been said above in comparing the two works to make further discussion of the first two volumes of Pascal's *Lezioni* superfluous. The third volume is little different from the first edition of 1897 of which the first part appeared in a German translation by Schepp in 1899.* It appears to be a reprint of this earlier volume rather than a new edition. It is this which accounts for the apparent disregard of the important developments in the calculus of variations during the last twenty years. It still frankly represents the old school, so that a detailed criticism from the modern point of view, for which the book furnishes ample opportunity, would clearly be out of place. The extensive bibliographies inserted at various points form a valuable feature. It is surprising, however, that in the list of treatises on page 19, apparently revised since the earlier edition, Hadamard's *Leçons* is not mentioned, while there is mention made of a *Lehrbuch der Variationsrechnung* by Carathéodory and Zermelo, which, although announced repeatedly, has not yet appeared and does not occur on Teubner's later lists of future publications.

ARNOLD DRESDEN

SHORTER NOTICES

Allgemeine Theorie der Raumkurven und Flächen. By V. Kommerell and K. Kommerell. Vols. I and II, third edition. (Sammlung Schubert, XXIX and XLIV.) Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1921. 184 + 196 pp. 28 + 13 figs.

The third edition of these two volumes is so nearly a reprint of the second edition, which has already been reviewed in this BULLETIN,† that an account of its contents is quite unnecessary. The arrangement of material is precisely the same in the two editions, but many of the discussions have been slightly amplified in the later one and the few errors in printing have been corrected. The only new material appears in the derivation of curves from given properties and in the definition of the Christoffel symbols.

These two volumes should be intelligible to the student who has little training beyond the calculus, yet they present an excellent treatment of the essentials of differential geometry. The student who has read them should have no difficulty with the more extensive treatments of Bianchi, Eisenhart, Forsyth and others. Such a presentation of differential geometry as this by Kommerell and Kommerell, if available in English, would increase the teaching of that important subject to the advanced students in our American universities.

E. B. STOUFFER

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* See the review by E. R. Hedrick in this BULLETIN, vol. 12 (1906), p. 172.

† Vol. 21 (1915), pp. 99–100.

Three Lectures on Fermat's Last Theorem. By L. J. Mordell. Cambridge, University Press, 1921. 30 pp.

The pamphlet consists of three lectures on Fermat's last theorem, given at Birkbeck College, London, in March 1920. The lectures, as stated in the preface, were intended primarily for persons with a mathematical training, but not necessarily for those who had made a special study of the theory of numbers. The work is divided into three chapters.

The first chapter contains a brief discussion of Fermat's work and the history of the theorem. It also contains the consideration of the early proofs of the impossibility of Fermat's equation $x^n + y^n = z^n$ for the cases $n = 3, 4, 5$, and 7.

The second chapter is an exposition of Kummer's work in attempting to prove the theorem. Algebraic numbers and the arithmetic in an algebraic domain are discussed, showing how the failure of the unique factorization in such a domain necessitates the introduction of ideal numbers. The nature of these ideal numbers is very clearly and briefly presented. The author explains in an elegant manner the main points in Kummer's work leading to the proof of the impossibility of the solution of $x^p + y^p = z^p$ in the domain defined by the primitive pth roots of unity, when the class number of the domain is prime to p. The chapter ends by a mention of the more recent results based on Kummer's work.

In the last chapter the author discusses the methods of Libri and Sophie Germain and the results obtained by these methods, of which may be mentioned Dickson's proof of the impossibility of a solution of Fermat's equation in integers prime to p (p the exponent) when $p < 7000$.

G. E. WAHLIN

The Absolute Relations of Time and Space. By Alfred A. Robb. Cambridge, University Press, 1921. 80 pp.

This is really a contribution to the geometry of point events in which stress is laid upon the importance of the ideas of before and after. To illustrate his ideas, the author makes frequent use of cones, somewhat after the manner of Minkowski, and introduces the idea of conical order. The reviewer believes that a reader will find it helpful in studying Robb's logical analysis if he also uses the geometric representation of a point event by a closed surface completely surrounding the point. An event A may then be regarded as after an event B if the representative surface of A completely surrounds the representative surface of B. If the two representative surfaces intersect or are external to one another, the event A is neither before nor after B. The geometry thus visualized may be regarded as a wave-geometry, the representative surface of an event A being the locus, at the very beginning of things, of point events whose combined influence results directly in the occurrence of the event A. The simplest assumption one can make is that the representative surface is a sphere with A as center, but a more general assumption is suitable for Einsteinian geometry and for the geometry of light waves in a material medium or of sound waves in a windy atmosphere.

H. BATEMAN

Philosophy and the New Physics. An Essay on the Relativity Theory and the Theory of Quanta. By L. Rougier. Authorized translation from the author's corrected text of *La matérialisation de l'énergie* by M. Masius. Philadelphia, Blakiston, 1921. 16mo. 16 + 159 pp.

The title of this translation is somewhat misleading. That of the original, *La Matérialisation de l'Energie*, gives a better idea of the scope of this work, which is an excellent presentation of the way in which the old dualistic view of matter and energy has given way to the modern merging of these two concepts into each other. The foundation for the modern view was laid when Sir J. J. Thomson showed, in 1881, that the inertia of an electrically charged sphere is increased by its motion. This led to the conception of electromagnetic momentum in the ether, and the author shows how the theory of relativity has solved the difficulties which this conception involves when it was combined with the stationary ether required for optical phenomena.

The book was apparently written before the author was familiar with the general theory of relativity, for Einstein's first value for the deflection of the light from a star by the sun's gravitational field is given. This is, however, corrected in a note later on. The author appears to be satisfied that the displacement of the solar spectral lines towards the red, demanded by the theory of relativity, has been experimentally confirmed, although this is far from being true.

Coming to the theory of quanta, it is difficult, as the author shows, to avoid the conclusion, if the consequences of the theory of relativity be accepted, that radiation is propagated by means of discrete elements or quanta. But until it is shown that the differential equations of optics, which are wholly successful in interpreting all ordinary optical phenomena, are consistent, perhaps only to a first approximation, with a quantum theory of radiation, one can hardly agree with the author in his extreme view.

There is a wrong reference to Poincaré's paper on the dynamics of the electron on page 73. The correct one is given in the bibliography on page 154.

E. P. ADAMS.

Das Exzentrizitätsprinzip als Korrelat zur Relativitätstheorie. By K. M. Kohler. Vienna, Franz Deuticke, 1921.

In this seventy-page pamphlet the author presents the restricted theory of relativity in a novel form by studying the geometry of the one-parameter family of spheres

$$(x - vt)^2 + y^2 + z^2 = c^2t^2, \qquad t > 0.$$

His conclusions are in accord with Einstein's theory and his method is of interest. He spoils an otherwise interesting paper, however, by claiming to have found a means of distinguishing between absolute motion and rest by correlating these two states with the eccentricity and concentricity of the electro-magnetic waves emitted by the body in question. Since he shows clearly that to an observer moving with a moving body the waves emitted by it would appear to be concentric, his claim is not to be treated seriously. This claim is not, however, organically related to the remainder of the paper.

C. N. REYNOLDS, JR.

NOTES

At the meeting of the International Astronomical Union, held at Rome, May 2–10, 1922, Dr. W. W. Campbell, of the Lick Observatory, was elected president for the coming three years. Professor Tuliio Levi Civita was appointed chairman of the committee on relativity.

The Jablonowski Society announces the following prize problem for 1924: In extending the Gaussian algorithm of the arithmetic-geometric mean to the case of two arbitrary functions $\varphi(x, y)$ and $\psi(x, y)$ two infinite sequences $x, x_1, \cdots, x_n, \cdots, y, y_1, \cdots, y_n, \cdots$ are obtained, starting from any two numbers x and y, by means of the recurrence formulas $x_n = \varphi(x_{n-1}, y_{n-1})$, $y_n = \psi(x_{n-1}, y_{n-1})$. The Society desires a discussion of such sequences for some simple functions $\varphi(x, y)$ and $\psi(x, y)$ for which an independent representation of the members of the sequences defined by the recurrence formulas can be found. Competing memoirs should be sent to the Jablonowski Society at the Library of the University of Leipzig before October 31, 1924; they should be written in German, Latin, or French.

Professor Henri Lebesgue, of the Sorbonne, has been elected a member of the Paris Academy of Sciences in the section of geometry, as successor to the late Camille Jordan.

Princeton University has conferred the honorary degree of doctor of science on Professor A. G. Webster, of Clark University.

Mr. W. W. Rouse Ball, of Trinity College, Cambridge, has offered to the University a sum of £500 to constitute a trust fund for the provision of occasional lectures dealing either with some particular development of mathematics or with some application of mathematics to science.

The Royal Society of Edinburgh has conferred its Keith prize (1919–1921) on Professor R. A. Sampson for his astronomical researches, including his papers on *Studies in clocks and time keeping*, published in its PROCEEDINGS during the period of the award. The same Society has conferred its James Scott prize, established in 1918 for a lecture or essay on the fundamental concepts of natural philosophy, on Professor A. N. Whitehead, for his lecture entitled *The relatedness of nature*.

The British commissioners of 1851 announce the following appointments to senior studentships in mathematics for 1922: Mr. A. E. Ingham, research student, Cambridge University; Mr. J. E. Jones, lecturer, University of Manchester. Holders of these studentships are given the opportunity of devoting their whole time for a period of not less than two years to scientific research.

The following advanced courses in mathematics are announced for the academic year 1922–23:

UNIVERSITY OF CHICAGO.—All courses meet four times a week for a quarter of twelve weeks. Courses which continue for more than one quarter are indicated with Roman numerals, as I, II, III, or IV.—By

Professor E. H. Moore: Vectors, matrices, and quaternions; Matrices in general analysis I, II, III, IV.—By Professor L. E. Dickson: Theory of numbers I, II; Solid analytics; Theory of equations.—By Professor H. E. Slaught: Differential equations; Elliptic integrals; Calculus I.—By Professor G. A. Bliss: Definite integrals; Elliptic functions; Calculus II, III.—By Professor E. J. Wilczynski: Projective differential geometry I, II; Functions of a complex variable; Calculus I, II.—By Professor F. R. Moulton: Analytic differential equations I, II, III; Advanced ballistics I, II, III.—By Professor W. D. MacMillan: Analytic mechanics I, II, III; Celestial mechanics.—By Professor A. C. Lunn: Units and dimensions; Dynamics of continuous media; Canonical equations and quantum theory; Thermodynamics.—By Dr. Mayme I. Logsdon: Theory of algebraic invariants; Calculus I, II, III.—By Professor J. W. A. Young: Limits and series. Courses in research are also offered by Professor Moore in Foundations of mathematics and in General analysis, by Professor Bliss in Analysis, by Professor Dickson in Algebra and Theory of numbers, by Professor Wilczynski in Geometry, and by Professor Lunn in Applied mathematics.

University of Illinois.—By Professor E. J. Townsend: Real variables.—By Professor G. A. Miller: Finite groups.—By Professor J. B. Shaw: Linear operators.—By Professor A. B. Coble: Differential geometry.—By Professor R. D. Carmichael: Linear differential equations in real variables.—By Professor A. Emch: Automorphic functions.—By Professor A. R. Crathorne: Theory of statistics.—By Professor A. J. Kempner: Modern algebra.—By Professor H. Blumberg: Introduction to higher mathematics.

Johns Hopkins University.—By Professor F. Morley: Higher geometry (first term); Theory of functions (second term).—By Professor A. Cohen: Applications of calculus, differential equations, and mechanics.—By Professor L. S. Hulburt: Advanced calculus; Projective geometry and higher plane curves.—By Dr. J. R. Musselman: Elementary theory of probability.—Professor Morley will conduct a seminary and a reading class.

Massachusetts Institute of Technology.—By Professor F. S. Woods: Advanced calculus.—By Professor C. L. E. Moore: Theoretical aeronautics.—By Professor H. B. Phillips: Thermodynamics.—By Professor J. Lipka: Analytical mechanics.—By Dr. N. Wiener: Fourier's series and integral equations.—By Dr. George Rutledge: Theory of functions.—By Dr. S. D. Zeldin: Vector analysis.—By Dr. J. S. Taylor: Mathematics of investment.

University of Michigan.—By Professor J. L. Markley: Solid analytic geometry (first term); Theory of functions of a complex variable; Theory of functions of real variables.—By Professor J. W. Glover: Theory of probability (first term); Finite differences (second term); Advanced mathematical theory of interest and life contingencies.—By Professor W. B. Ford: Advanced calculus, with especial reference to Fourier series and harmonic analysis; Infinite series and products; Elements of the calculus of variations (first term).—By Professor L. C. Karpinski: Higher algebra; Theory of numbers; History of mathematics.—By Professor

J. W. Bradshaw: Introduction to modern geometry (second term); Projective geometry.—By Professor R. B. Robbins: Casualty actuarial theory. —By Mr. R. W. Barnard: Differential equations (first term); Mathematical theory of statistics, advanced course.—By Professor A. Ziwet: Hydrodynamics.—By Professor P. Field: Projective geometry for engineers (first term); Vector analysis (second term).—By Professor T. R. Running: Graphical methods (first term); Empirical formulas (second term); Advanced calculus (first term).—By Professor T. E. Hildebrandt: Theory of the potential (first term).—By Professor V. C. Poor: Theoretical mechanics.—By Professor L. J. Rouse: Fourier series (second term).

University of Pennsylvania.—By Professor E. S. Crawley: Modern analytic geometry (first term); Differential equations (first term); Higher plane curves (second term).—By Professor G. H. Hallett: Infinite series and products (first term); Functions of a complex variable (second term). —By Professor H. B. Evans: Quaternions and vector methods (second term).—By Professor O. E. Glenn: Calculus of variations.—By Professor F. H. Safford: Mathematical theory of elasticity.—By Professor G. G. Chambers: Introduction to higher algebra.—By Professor H. H. Mitchell: Linear groups (first term); Advanced calculus (second term).—By Professor M. J. Babb: History of mathematics.—By Professor F. W. Beal: Differential geometry.—By Professor J. R. Kline: Foundations of mathematics (first term); Continuous transformations (second term).

University of Wisconsin.—By Professor E. P. Lane: Modern analytical geometry.—By Professor E. B. Van Vleck: Functions of a real variable; Integral equations.—By Professor H. W. March: Theoretical hydrodynamics.—By Professor C. S. Slichter: Potential theory.—By Professor E. B. Skinner: Higher algebra.—By Professor A. Dresden: Calculus of variations.

Yale University.—By Professor E. W. Brown: Mechanics; Advanced mechanics; Hydromechanics.—By Professor J. Pierpont: Functions of a complex variable; Projective and differential geometry; Approximation methods.—By Professor P. F. Smith: Differential equations.—By Professor W. A. Wilson: Theory of aggregates.—By Professor E. J. Miles: Advanced calculus; Calculus of variations.—By Professor J. I. Tracey: Advanced analytic geometry.—By Professor W. L. Crum: Mathematical statistics.—By Professor J. K. Whittemore: Advanced differential geometry.

Professor A. Sommerfeld, professor of mathematical physics at the University of Munich, will be in residence at the University of Wisconsin for the first semester of the coming academic year. He will hold during that period the Karl Schurz Memorial Professorship, which was established at the University of Wisconsin in 1910, and is resumed with this appointment after the interruption caused by the war. Professor Sommerfeld is expected to offer a three-hour course on Atomic structure, and a three-hour course either on the Analysis of wave propagation, or on the General theory of relativity. Professor Sommerfeld is known to American scholars in pure mathematics as well as to those in mathematical physics.

Professor L. Tonelli, of the University of Parma, has been appointed professor of higher analysis at the University of Bologna.

Professor L. Silla, of the University of Cagliari, has been appointed professor of rational mechanics at the University of Genoa.

Professor E. Bertini, of the University of Pisa, has retired from active service.

At the University of Lille, Professor A. Chatelet has been transferred from the professorship of general mathematics to that of rational mechanics; Dr. Gambier succeeds him as professor of general mathematics.

At Cambridge University, Mr. E. A. Milne, of Trinity College, has been appointed University lecturer in astrophysics, and Mr. S. Lees, of St. John's College, University lecturer in thermodynamics.

Professor W. R. Burwell, of the University of Tennessee, has been appointed assistant professor of mathematics and dean of freshmen at Brown University.

Professor K. D. Swartzel, formerly of the University of Ohio, has been appointed head of the department of mathematics at the University of Pittsburgh.

Miss Helen Barton, of Wellesley College, has been appointed dean of women at Albion College.

Mr. W. H. Hill, of the University of Colorado, has been appointed assistant professor of mathematics at the Manual Training Normal School, Pittsburg, Kansas.

Mr. H. K. Cummings has resigned his instructorship in mathematics at Brown University to accept a position in the Bureau of Standards at Washington.

At Iowa State College, Associate Professor J. T. Colpitts has been granted leave of absence for the coming year to study at Cornell University; Assistant Professor E. C. Kiefer has been granted leave of absence for the coming year to study at the University of Michigan; Dr. H. C. Gossard, formerly of the University of Oklahoma and the U. S. Naval Academy, has been appointed to an assistant professorship.

Professor W. J. Risley has resigned as professor of mathematics at James Millikan University, where he has been since 1910, to accept a position as head of the department of mathematics at the Colorado School of Mines.

Professor William Marshall, who has been acting head of the department of mathematics at Purdue University during the past year, has been appointed head of the department there.

At the California Institute of Technology, Professor Harry Bateman, formerly professor of aeronautical research and mathematical physics, has been made professor of mathematics, theoretical physics, and aeronautics; Dr. Clyde Wolfe, formerly instructor, has been made assistant professor of mathematics; Dr. R. C. Tolman has been appointed professor of physical chemistry and mathematical physics.

At the University of Manitoba, Mr. F. S. Nowlan has been promoted to an assistant professorship of mathematics.

Professor J. D. Bond, of Louisiana State University, has been appointed associate professor of mathematics at the University of Tennessee.

At the University of Colorado, Assistant Professor G. H. Light has been promoted to a full professorship and Dr. Claribel Kendall to an assistant professorship.

At the University of Washington, Associate Professor E. T. Bell has been promoted to a full professorship and Dr. L. L. Smail to an assistant professorship. Dr. Hermance Mullemeister has been granted a year's leave of absence for study in Holland; her place will be taken by Miss E. D. Pepper.

Assistant Professor R. B. Robbins will return to the University of Michigan after two years' absence in actuarial work in the departments of insurance of Missouri and New York.

At the University of California, Professor C. A. Noble has been granted a term's leave of absence, which he will spend in Europe.

At the Georgia School of Technology, Professor B. M. Boerckel has resigned and Mr. R. M. Mundorff has been appointed assistant professor.

At Dartmouth College, Assistant Professor F. M. Morgan has resigned.

The following appointments to instructorships at American colleges and universities are announced:

Brown University, Mr. J. H. Fithian, Mr. H. C. Hicks, Mr. A. O. Hickson;

University of California, Dr. P. H. Daus (transferred to southern branch), Dr. B. C. Wong;

Carnegie Institute of Technology, Mr. George Parks;
Dartmouth College, Dr. B. H. Brown;
Georgia School of Technology, Mr. H. K. Fuhner;
Hood College, Miss M. C. Packer;
University of Iowa, Mr. R. E. Kennon;
Iowa State College, Mr. R. G. Robinson;
University of Michigan, Mr. R. V. Churchill, Mr. C. C. Craig;
University of Missouri, Mr. C. G. Jaeger;
University of Nebraska, Mr. F. S. Harper;
New York University, Mr. F. W. John;
University of Pennsylvania, Mr. J. M. Thomas;
Pennsylvania State College, Mr. R. H. Marquis;
Purdue University, Dr. W. E. Edington, Mr. W. J. Wagner;
Randolph-Macon College, Mr. J. W. Blincoe;
Smith College, Miss Constance Wiener;
Syracuse University, Miss May Sperry;
Wellesley College, Miss F. M. Merriam;
University of Wisconsin, Mr. E. B. Keller, Mr. A. H. Wait.

Professor O. Tedone, of the University of Genoa, died April 18, 1922, at the age of fifty-two years.

Professor Hermann Grassmann, of the University of Giessen, died in January, 1922, at the age of sixty-five years.

Professor J. C. Kapteyn, professor of astronomy and mechanics at the University of Groningen, died June 18, 1922.

Professor Pierre Boutroux, of the Collège de France, died June 25, 1922. Professor Boutroux had been a member of the American Mathematical Society since 1913.

Professor James McMahon, of Cornell University, died June 1, 1922, at the age of sixty-six years. Professor McMahon had been a member of the American Mathematical Society since its organization (as the New York Mathematical Society) in 1891. He was for seven years associate editor of the Annals of Mathematics, and was secretary of section A, general secretary, and vice-president of the American Association for the Advancement of Science.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

BENNY (L. B.). Plane geometry. An account of the more elementary properties of the conic sections, treated by the methods of co-ordinate geometry, and of modern projective geometry, with applications to practical drawing. London, Blackie, 1922. 8 + 336 pp.

BIEBERBACH (L.). Funktionentheorie. (Teubners Technische Leitfäden, Band 14.) Leipzig, Teubner, 1922. 118 pp.

BOREL (E.). See LÉVY (P.).

CARTAN (E.). Leçons sur les invariants intégraux. Paris, Hermann, 1922. 10 + 210 pp.

DENJOY (A.). Calcul des coefficients de la série trigonométrique convergente la plus générale dont la somme est une fonction donnée. Paris, Gauthier-Villars, 1921. 4to. 16 pp.

GOURSAT (E.). Leçons sur le problème de Pfaff. Paris, Hermann, 1922. 8 + 387 pp.

HAAG (J.). Cours complet de mathématiques spéciales. Tome 2: Géométrie. Paris, Gauthier-Villars, 1921. 8vo. 6 + 662 pp.

HADAMARD (J.). See JUVET (G.), LÉVY (P.).

JUVET (G.). Introduction au calcul tensorial et au calcul différentiel absolu. Préface de M. J. Hadamard. Paris, Blanchard, 1922. 8vo. 2 + 101 pp.

KOWALEWSKI (G.). Die klassischen Probleme der Analysis des Unendlichen. Ein Lehr- und Uebungsbuch für Studierende zur Einführung in die Infinitesimalrechnung. 2te Auflage. Leipzig, Engelmann, 1921. 8vo. 8 + 342 pp.

LÉVY (P.). Leçons d'analyse fonctionnelle. Avec une préface de M. J. Hadamard. (Collection de Monographies sur la Théorie des fonctions, publiée sous la direction de M. E. Borel.) Paris, Gauthier-Villars, 1922. 6 + 442 pp.

LINDOW (M.). Differentialgleichungen. (Aus Natur und Geisteswelt, Nr. 589.) Leipzig, 1921.

POST (J. F.). Ueber die Darstellung ganzer Zahlen als Summen von sieben Kuben. (Diss., Halle-Wittenberg.) Halle, Hohmann, 1920. 36 pp.

RIEHLE (A.). Ueber den Bertinischen Satz und seine Erweiterung. (Diss.) Tübingen, 1920.

SIMON (P.). La recherche des lieux géométriques en géométrie analytique. Paris, Colin, 1922. 8vo. 232 pp.

UHLER (A.). Sur les séries zétafuchsiennes. (Diss., Lund.) Lund, 1921. 57 pp.

II. ELEMENTARY MATHEMATICS

AUGUST (E. F.). Vollständige logarithmische und trigonometrische Tafeln. Berlin, 1922.

BARTHOLOMEW (W. E.). See EDGERTON (E. I.).

EDGERTON (E. I.) and BARTHOLOMEW (W. E.). Business mathematics. New York, Ronald Press Company, 1921. 6 + 305 pp.

FORD (W. B.). A brief course in college algebra. New York, Macmillan, 1922.

KEAL (H. M.) and LEONARD (C. J.). Mathematics for shop and drawing students. New York, Wiley, 1921.

LEONARD (C. J.). See KEAL (H. M.).

PASSANO (L. M.). Calculus and graphs simplified for a first brief course. New York, Macmillan, 1921. 8 + 167 pp.

SICELOFF (L. P.), WENTWORTH (G.) and SMITH (D. E.). Analytic geometry. Brief course. Boston, Ginn, 1922. 6 + 186 pp. $1.80

SMITH (D. E.). See SICELOFF (L. P.).

WENTWORTH (G.). See SICELOFF (L. P.).

III. APPLIED MATHEMATICS

BACH (C.). Elastizität und Festigkeit. 8te, vermehrte Auflage. Unter Mitwirkung von R. Baumann. Berlin, Springer, 1920.

BAUMANN (R.). See BACH (C.).

BAVINK (B.). Grundriss der neueren Atomistik. Leipzig, Hirzel, 1922. 8vo. 6 + 130 pp.

BELLENOT (H.). See BORN (M.).

BLANC (A.). Rayonnement. Principes scientifiques de l'éclairage. Paris, Colin, 1921. 16mo. 212 pp.

BOREL (E.). L'espace et le temps. Paris, Alcan, 1922. 16mo. 245 pp.

BORN (M.). La constitution de la matière. Traduit par H. Bellenot. Paris, Blanchard, 1922. 8vo. 84 pp.

BROMWICH (T. J. I'a.). Examples in optics. Cambridge, Bowes and Bowes, 1921. 16 pp.

CASE (J.). Notes and examples on the theory of heat and heat engines. Cambridge, Heffer, 1922.

Clarke (H. T.). See Planck (M.).

Clibbens (D. A.). The principles of the phase theory. Heterogeneous equilibria between salts and their aqueous solutions. London, Macmillan, 1920. 20 + 382 pp. 25s.

Cowley (W. L.) and Levy (H.). Aeronautics in theory and experiment. 2d edition. London, Arnold, 1920. 12 + 331 pp. 25s.

Dubroca (M.). Au sujet de la théorie de la relativité restreinte. 1e partie. Dijon, Imprimerie R. de Thorey, 1921. 8vo. 76 pp.

Emsley (H. H.). See Gleichen (A.).

Föppl (A.). Vorlesungen über technische Mechanik. Band 1: Einführung in die Mechanik. 7te Auflage. Band 2: Graphische Statik. 5te Auflage. Band 3: Festigkeitslehre. 8te Auflage. Band 4: Dynamik. 6te Auflage. Leipzig, 1920–1921.

Geiger (H.) und Makower (W.). Messmethoden auf dem Gebiete der Radioaktivität. Braunschweig, Vieweg, 1920. 9 + 156 pp.

Gleichen (A.). The theory of modern optical instruments. Translated from the German by H. H. Emsley and W. Swaine. 2d edition. London, H. M. Stationery Office, 1921. 11 + 365 pp.

Glover (J. W.). See United States.

Grammel (R.). Die mechanischen Beweise für die Bewegung der Erde. Berlin, Springer, 1922. 2 + 71 pp.

Haldane (Viscount). La règne de la relativité. Traduction française de H. de Varigny. Paris, Gauthier-Villars, 1922. 8vo. 590 pp.

Higgins (A. L.). The transition spiral and its introduction to railway curves. With field exercises in construction and alignment. London, Constable, 1921. 8 + 111 pp. 6s.

von Ignatowsky (W.). Die Vektoranalysis und ihre Anwendung in die theoretischen Physik. 1ter Teil. 2te Auflage. Leipzig, 1921.

Juvet (G.). See Weyl (H.).

Laurent (M.). Les trois états de matière. Etat fluide, état corpusculaire, corps simples. Paris, Maloine, 1922. 8vo. 52 pp.

Lecornu (L.). Dynamique appliquée. 2e édition, revue, corrigée et augmentée. 2 volumes. Paris, Doin, 1921.

Lenard (P.). Ueber Relativitätsprinzip. Aether, Gravitation. Neue, vermehrte Ausgabe. Leipzig, Hirzel, 1920.

Leroy (R.). See Weyl (H.).

Levy (H.). See Cowley (W. L.).

de Losada y Puga (C.). Las anomalias de la gravidad. Lima, Torres Aguirre, 1920.

Lotze (A.). Die Grundgleichungen der Mechanik insbesondere starrer Körper, neu entwickelt mit Grassmanns Punktrechnung. Leipzig, Teubner, 1922. 8vo. 50 pp.

McCabe (J.). See Nordmann (C.).

Maggi (G. A.). Elementi di statica e teoria dei vettori applicati. Pisa, Spoerri, 1920. 8vo. 176 pp.

Makower (W.). See Geiger (H.).

Miall (S.). The structure of the atom. Notes on some recent theories. London, Benn, 1922. 3 + 26 pp.

Mills (J.). Within the atom. Popular view of elektrons and quanta. New York, Van Nostrand, 1921. 232 pp. $2.25

Nordmann (C.). Einstein and the universe. A popular exposition of the famous theory. Translated by J. McCabe. London, Fisher Unwin, 1922. 185 pp.

Oppenheim (S.). Das astronomische Weltbild im Wandel der Zeit. I: Vom Altertum bis zur Neuzeit. 3te Auflage. Leipzig, Teubner, 1920.

Philip (A.). The calendar. Its history, structure, and improvement. Cambridge, University Press, 1921. 12 + 104 pp. 7s. 6d.

Pillet (J.). Traité de géométrie descriptive. Nouveau tirage. Paris, 1921. 4to. 270 pp.

Planck (M.). Einführung in die Theorie der Elektrizität. Leipzig, 1922.

——. The origin and development of the quantum theory. Translated by H. T. Clarke and L. Silberstein. Oxford, Clarendon Press, 1922. 8vo. 23 pp.

Reade (W. H. V.). A criticism of Einstein and his problem. Oxford, Blackwell, 1922. 6 + 126 pp.

von Rohr (M.). Theorie der optischen Instrumente. Berlin, 1922.

Russell (R. A. B.). See Thirring (J. H.).

Schmidt (H.). Das Weltbild der Relativitätstheorie. 2te, vermehrte Auflage. Hamburg, 1920.

Silberstein (L.). See Planck (M.).

Sommerfeld (A.). Atombau und Spektrallinien. 3te Auflage. Braunschweig, Vieweg, 1922. 8vo.

Swaine (W.). See Gleichen (A.).

Tammann (G.). Aggregatzustände. Die Zuständigkeitsveränderungen der Materie von Druck und Temperatur. Leipzig, Voss, 1922. 8 + 294 pp.

Thirring (J. H.). The ideas of Einstein's theory. A theory of relativity in simple language. Translated by R. A. B. Russell. London, Methuen, 1921. 15 + 167 pp. 5s.

Timerding (H. E.). Die Fallgesetze. 2te Auflage. Leipzig, Teubner, 1921. 4 + 51 pp.

United States life tables, 1890, 1901, 1910, and 1901–1910. Explanatory text, mathematical theory, computations, graphs, and original statistics, also tables of United States life annuities, life tables of foreign countries, mortality tables of life insurance companies. Prepared by J. W. Glover. Washington, Government Printing Office, 1921. 4to. 496 pp.

de Varigny (H.). See Haldane (Viscount).

Velasco de Pando (M.). Calcolo de las probabilides. Paris, Gauthier-Villars, 1920. 8vo. 430 pp.

Villey (J.). Physique élémentaire et théories modernes. I: Molécules et atomes. Paris, Gauthier-Villars, 1921. 8vo. 10 + 198 pp.

Weyl (H.). Temps, espace, matière. Leçons sur la théorie de la relativité générale. Traduit sur la 4e édition allemande par G. Juvet et R. Leroy. Paris, Blanchard, 1922. 8vo. 8 + 290 pp.

CREMONA TRANSFORMATIONS AND APPLICATIONS TO ALGEBRA, GEOMETRY, AND MODULAR FUNCTIONS *

BY A. B. COBLE

1. *Introduction.* Two of the most highly developed fields of modern mathematics are those associated with the projective group and the birational group. We have on the one hand projective geometry with its analytic counterpart in the theory of algebraic forms, and on the other hand algebraic geometry and algebraic functions. Between these two domains there lies the group of Cremona transformations for which as yet no distinctive geometry and no distinctive invariant theory has been formulated. It seems opportune therefore to give this brief résumé of achievement in this field along the somewhat scattered lines in which research has been pursued, to indicate certain problems that await solution, and to point out certain directions in which results of importance may be expected.

Several topics are omitted which perhaps first occur to one's mind when Cremona transformations are mentioned. The most important of these omissions is the quadratic transformation. This, first introduced analytically by Plücker(1)† in 1829 and by Magnus(2) in 1831 as a reciprocal radii transformation, has been the subject of repeated investigation down to recent papers of Emch(3). It is for the geometer a powerful instrument for simplifying a problem or for extending a theorem.

The group of plane motions underlying euclidean geometry may be enlarged in one direction to the projective group and projective geometry or in another direction to the inversive group and inversive geometry. The quadratic transformations of this latter group were first considered in general by Möbius (4) in 1853 under the name of *Kreisverwandschaften.*

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* Presented before the Society at the Symposium held in Chicago, April 14, 1922.

† Such numbers refer to the papers listed at the close of this article.

Ten years later, in 1863, Luigi Cremona(5) first established the theory of the general birational point transformation valid throughout the plane. This was extended to space in 1869–71 by Cayley(6), Cremona(7), and Noether(8).

Before taking up this general theory two famous theorems deserve mention. The first states that the general ternary Cremona transformation is a product of quadratic transformations. This was surmised by Clifford(9) in 1869, and was verified by Cayley(10). Imperfect proofs given by Noether(11) and Rosanes(12) in 1870 were finally replaced in 1901 by a rigorous proof due to Castelnuovo(13). Unfortunately this theorem has no analog in space, as I shall point out later. The second theorem, proved by Noether(14) in 1871, asserts that any algebraic curve can be transformed by a Cremona transformation into a curve with isolated multiple points with distinct tangents. The curve as thus simplified is a suitable basis for the algebraic functions which Noether had in mind. A proof more geometric in character was given by Bertini(15) in 1888.

A. CREMONA TRANSFORMATIONS FROM SPACE TO SPACE

2. *Classification of Algebraic Correspondences.* This section is largely introductory. Consider the rational transformation*

$$(1) \qquad \begin{cases} y_0 = f_0(x_0, x_1, x_2)^n, \\ y_1 = f_1(x_0, x_1, x_2)^n, \\ y_2 = f_2(x_0, x_1, x_2)^n, \end{cases}$$

from point x of a plane E_x to point y of a plane E_y, where, first, the three curves f of order n do not belong to a pencil,† and second, the three curves f are not compounded of members of a pencil, as would be the case, for example, if they were pairs of lines on a point. In short the jacobian $J(f_0, f_1, f_2)$ does not vanish identically. Then, as x describes the plane E_x, y describes the plane E_y and *vice versa.* We have thereby merely established a projective correspondence between the net of lines $\eta_0 y_0 + \eta_1 y_1 + \eta_2 y_2 = 0$ in E_y and the proper net of curves $\eta_0 f_0 + \eta_1 f_1 + \eta_2 f_2 = 0$ in the plane E_x, where the term proper implies that $J \neq 0$. To the point y, the base of a line pencil in E_y, there corresponds in E_x the variable base points of the corresponding pencil of curves in E_x.

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* Throughout this paper, homogeneous coordinates are used.

† I shall use the terms pencil, net, and web for a linear system of ∞^1, ∞^2, ∞^3 things respectively.

We distinguish here three cases. First, the case $n = 1$, for which the net in E_x is merely the net of lines whose pencils have a single base point variable with the pencil. This is the familiar *projective transformation.* Second, the case in which $n > 1$ and in which the pencils of the net still have a single base point on E_x variable with the pencil. This is the general *Cremona transformation.* The coordinates of the variable base point can be expressed rationally in terms of those of y, and the transformation is birational throughout the planes. Third, the case in which $n > 1$ and in which the pencils of the net have k base points variable with the pencil. This is a 1 to k correspondence between E_y and E_x, and the coordinates x are irrational k-valued functions of the coordinates y. The birationality is restored by a simple device. As x runs over a curve $g(x) = 0$ in E_x, y runs over a curve $h(y) = 0$ in E_y; and, for each position of y on $h(y)$, in general only one of the k corresponding points x on E_x will lie on the given curve $g(x)$, so that equations (1) together with $g(x) = 0$ establish a birational correspondence between x and y which, however, is limited to the curves g, h. To such transformations, one-to-one over limited regions of the linear spaces in question, the term *birational transformation* will be confined.

If to (1) we add new equations of the same form, say,

$$(2) \qquad \begin{cases} y_3 = f_3(x_0, x_1, x_2)^n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_d = f_d(x_0, x_1, x_2)^n, \end{cases}$$

thereby enlarging the net of curves f to a linear system of dimension d and expanding the plane E_y to the linear space S_d, then we have a *mapping* of the plane E_x upon a manifold $M_2{}^k$ of dimension 2 and order k in S_d, which in general is birational.

3. *Fundamental Points and Principal Curves.* Reverting to the case of the Cremona transformation (1) for which a single base point x of the pencils of the net is variable, we can show(16) that the fixed base points of the pencils are fixed for the entire net, say at $p_1, p_2, \cdots, p_\rho$ with multiplicities $r_1, r_2, \cdots, r_\rho$ respectively. For these points—the *fundamental points,* or *F-points,* of the transformation—the functions f vanish and y is indeterminate. For simplicity I shall assume that these points are isolated. It is, however, a single linear condition on the net that a curve of the net shall pass through an F-point p of multiplicity r with a given direction. To the pencil of curves with this given direction there corresponds on E_y a pencil of lines on a definite point y. The locus of such points y is the *principal curve,* or *P-curve,* on E_y which corresponds to p on E_x, or rather to the directions about p on E_x. It is rational and of order r, since a line on E_y meets it in as many points as the corresponding curve of the net on E_x has directions at p. The P-curves on E_y can meet only at the F-points on E_y of the inverse transformation, say $q_1, q_2, \cdots, q_\sigma$ of multiplicities $s_1, s_2, \cdots, s_\sigma$ ($\sigma = \rho$(17)) for the curves of the net on E_y (also of order n(17)) which correspond to the lines on E_x. To the directions about an F-point on E_y there corresponds a P-curve on E_x which can meet the general curve of the net only at the base points. It must therefore be a fixed rational constituent of a pencil of the net. Conversely if a pencil of the net has a fixed part, this part must correspond to a single point,—necessarily an F-point,—on E_y. Thus if the net (1) is given on E_x, and its base points of multiplicities r are thereby known, the pencils with fixed parts are easily isolated. If also the projective correspondence between lines on E_y and curves of the net is given, the pencils with fixed parts determine the position of the F-points on E_y and the orders of the fixed parts are the multiplicities of these F-points for the net on E_y. The number of constants required to determine the transformation is $2\rho + 8$. If α_{ij} is the number of times the curve P_j on E_y, which corresponds to p_j on E_x, goes through q_i on E_y, it is the number of directions at q_i which correspond to direc-

tions at p_j, and therefore also the number of times the curve P_i on E_x, which corresponds to q_i, passes through p_j. We shall have occasion to use these numbers soon.

Cremona transformations in space are obtained by setting up a projective correspondence between the web of planes in $S_3(y)$ and a *homoloidal* web of surfaces in $S_3(x)$, i.e., a web such that the nets of surfaces in the web have a single variable intersection x which describes completely the space $S_3(x)$. We now find F-points of three kinds which are exemplified by the cubi-cubic transformation determined by three bilinear forms

$$(\alpha_i x)(\beta_i y) = 0, \qquad (i = 1, 2, 3). \tag{3}$$

Here for general forms a point x determines three planes which meet in a unique point y unless x happens to be on the sextic curve defined by the vanishing of the (3, 4) matrix of coefficients of y. Then the three planes meet in a line every point y of which is a correspondent of the given point x. We say then that x is an F-point of an F-curve *of the first kind* and that its corresponding line is a P-curve whose locus is a *P-surface of the first kind.* The ∞^2 directions at x lie ∞^1 at a time in the ∞^1 planes on the tangent line to the F-curve at x, and all the directions at x in one of these planes correspond to a single point on the P-curve of x.

However, in the particular case when the forms (3) are

$$\textstyle\sum_{j=0}^{j=3} a_{ij}x_jy_j = 0, \qquad (i = 1, 2, 3), \tag{4}$$

the transformation just considered degenerates into the familiar type,

$$y_j = r_j/x_j = \rho r_j x_k x_l x_m, \quad (j, k, l, m = 0, 1, 2, 3). \tag{5}$$

The sextic F-curve has become the six edges of a tetrahedron T_x. The point x on such an edge is still an F-point. The ∞^2 directions about it correspond no longer to points on a P-curve, but rather to directions about a definite point on an edge of a tetrahedron T_y in $S_3(y)$. We say that the six edges are *F-curves of the second kind.* They have no corresponding P-surfaces. Another novelty here is the four *F-points of the second kind,* the four vertices of T_x. To the ∞^2 directions about one of these there correspond the ∞^2 points on a plane of T_y so that to an F-point of the second kind there corresponds a P-surface. Moreover the F-curves of the second kind are a necessary consequence of the existence of the F-points of the second kind, since the cubic surfaces of the web with nodes at the vertices of T_x must necessarily contain the edges of T_x. This space transformation is the immediate extension of the quadratic transformation of the plane.

In four dimensions, a Cremona transformation may have F-surfaces, F-curves, and F-points, each of various kinds, and obviously the possible complications increase with the dimension. However, the transformation of the type (5) preserves its form throughout, being determined in $S_k(x)$, $S_k(y)$ by a set of $k+1$ F-points in either space and a pair of corresponding points, all other F-points of the transformation being a necessary consequence of the existence of the given sets.

To replace the theorem which states that in the plane direct and inverse transformations have the same number of F-points, Pannelli(18) proves that, if in space they have respectively σ, σ' F-points and τ, τ' F-curves of genera ρ_i, ρ_i', then

$$\sigma + \tau - \Sigma_i \rho_i = \sigma' + \tau' - \Sigma_i \rho_i'.$$

From the fact that in space the product of two transformations with respectively F-curves of genera ρ_1, ρ_2 will itself have F-curves of genera equal to both ρ_1 and ρ_2, and the further fact that transformations can be constructed with an F-curve of arbitrarily great genus Miss Hilda Hudson(19) concludes that the general space transformation can not be expressed as a product of a finite number of given types.

4. *Types of Cremona Transformation.* If the Cremona transformation of order n in the plane has F-points of multiplicities $r_1, r_2, \cdots, r_\rho$ arranged in descending order of magnitude, we deduce from the fact that the curves of the net are rational, and that they have but a single variable intersection, the necessary conditions

$$(6)\qquad \begin{cases} r_1^2 + r_2^2 + \cdots + r_\rho^2 = n^2 - 1, \\ r_1 + r_2 + \cdots + r_\rho = 3(n-1), \end{cases}$$

on the positive integers $n, \rho, r_1, \cdots, r_\rho$. In addition to these there are an unlimited number of inequalities of the form

$$(7)\qquad \left\{\begin{array}{l} r_1 + r_2 \qquad\qquad\qquad \leqq \; n, \\ r_1 + \qquad \cdots + r_5 \leqq 2n, \\ 2r_1 + r_2 + \cdots + r_7 \leqq 3n, \\ r_1 + \qquad \cdots + r_9 \leqq 3n, \text{ etc.,} \end{array}\right.$$

which must be satisfied by these integers lest the curves of the net all contain a factor of order 1, or 2, or 3, etc. For given n, the solutions of this Diophantine system are finite in number and the tabulations of such solutions from Cayley's(20) articles to the more recent articles of Montesano(21) and Larice(22) are arranged in this way. It will appear, I hope, that a more natural classification of these solutions is according to the number ρ of F-points. For $\rho < 9$ the number of solutions is finite. For $\rho = 9$ the number is infinite and I have proved(23) that this infinite number can be arranged in $960 \cdot 3^7 \cdot 2$ classes, such that each class contains an infinite number of solutions depending on the unrestricted variation of eight integers. This is by far the most extensive aggregate of solutions as yet formulated.

The transformation (1) transforms a curve of order μ_0 with multiplicities $\mu_1, \cdots, \mu_\rho$ at the F-points on E_x into a curve of order μ_0' with multiplicities $\mu_1', \cdots, \mu_\rho'$ at the F-points on E_y, where

$$(8)\qquad \left\{\begin{array}{l} \mu_0' = n\mu_0 - r_1\mu_1 - r_2\mu_2 \cdots - r_\rho\mu_\rho, \\ \mu_1' = s_1\mu_0 - \alpha_{11}\mu_1 - \alpha_{12}\mu_2 \cdots - \alpha_{1\rho}\mu_\rho, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \mu_\rho' = s_\rho\mu_0 - \alpha_{\rho_1}\mu_1 - \alpha_{\rho_2}\mu_2 \cdots - \alpha_{\rho\rho}\,\mu_\rho. \end{array}\right.$$

If also pairs of ordinary points are to be considered we add

$$(9)\qquad \mu_{\rho}{}'_{+1} = \qquad\qquad -(-1)\mu_{\rho+1}, \text{ etc.}$$

The coefficients n, r_j, s_i, α_{ij} of this linear transformation determine the *type*(23) of the Cremona transformation. All types of this kind can be generated by that of the quadratic transformation, namely,

$$(10)\qquad \left\{\begin{array}{l} \mu_0' = 2\mu_0 - \mu_1 - \mu_2 - \mu_3, \\ \mu_1' = \mu_0 \qquad\quad - \mu_2 - \mu_3, \\ \mu_2' = \mu_0 - \mu_1 \qquad\quad - \mu_3, \\ \mu_3' = \mu_0 - \mu_1 - \mu_2, \end{array}\right.$$

together with permutations of the μ's or μ''s. In this way, *for given* ρ, we obtain a linear group with integral coefficients whose elements furnish the solutions of (6), (7), a group whose general modular theory has yet to be studied.

For spaces of higher dimension, we define a *regular* Cremona transformation to be a product of projectivities, and of a single transformation of the type (5), $y_i' = 1/x_i$, $(i = 0, 1, \cdots, k)$. Such regular transformations have properties entirely analogous to those of the plane. For S_3 they have been studied by S. Kantor(24) and for S_k by myself(23). The preceding arithmetic discussion is generalized in my paper(23).

B. CREMONA TRANSFORMATIONS IN A SINGLE SPACE

5. *The Cremona Group and its Subgroups.* In this section the planes E_x and E_y are understood to be superposed. We should perhaps speak rather of the Cremona transformation from point x to point x' of the same plane. The totality of such transformations form a group. This group does not depend on a finite number of parameters since an element with ρ F-points depends upon $2\rho + 8$ constants and ρ may be as large as we please. Nor is it an infinite group in the sense of Lie, since it is not defined by differential equations. Though it contains continuous subgroups its most striking property is the *discontinuity* which appears in the variation of the order n of its elements throughout the range of positive integers. One might expect therefore to find within or associated with it a rich array of discontinuous groups both finite and infinite.

Enriques(25) has shown that any finite continuous subgroup in the plane can be transformed by Cremona transformation into one of three types: (a) the 8-parameter collineation group; (b) the 6-parameter group of inversions; and (c) the *Jonquière groups* J_m, which carry into itself a pencil of rays on a point O, and also a linear system of curves of order m with an $(m - 1)$-fold point at O and common tangents at O; or into a continuous subgroup of one of these three types. Enriques and Fano(26) have made a similar study for space with analogous results. Noether(27) has classified continuous groups of quadratic transformations in S_3 into five total groups with various subgroups. In general one may say that any continuous group of Cremona transformations in S_k can be regarded as a projective group in a space S_d, $d > k$, which has an invariant M_k. For the group has an invariant linear system of dimension d which maps S_k upon an M_k in S_d. Mohrmann(28) has determined these surfaces in S_d for the three types of Enriques.

Apart from the projective subgroup, the most important of the groups of Enriques is the inversive group. If the F-points of a quadratic transformation and its inverse are respectively (p_1, p_2, p_3) and (q_1, q_2, q_3), such that directions at p_i correspond to points on q_jq_k, the group of inversions is made up of those quadratic transformations for which the pairs (p_1, p_2) and (q_1, q_2) coincide at the circular points. The applications of this group in metric geometry and function-theory are well known. It may be extended to space in various ways. If the circular points be replaced by the absolute conic at infinity, the group of quadratic inversions in space is obtained. If the pair of circular points be replaced by three lines in space, a quite different group of cubic transformations appears. Young and Morgan(29) have discussed this group and its extensions with particular reference to the cubic curves bisecant to the three lines. These curves are *one* extension of the circle in the plane; but perhaps a more natural extension is to the cubic surfaces containing the three lines. Such surfaces are given analytically by trilinear binary forms as circles are given by bilinear binary forms in z, $\bar{z}$.

Mention should be made at this point of infinite discontinuous subgroups of the Jonquière type. If in the plane we fix (p_1, q_1) at O, the quadratic transformations generate the group of all Jonquière transformations of any order n with $(n - 1)$-fold F-point (both direct and inverse) at O. This group has as an invariant the line pencil at O. It is of the same general character as the Cremona group itself. If in space we consider the regular transformations generated by cubic transformations with F-tetrahedra $(p_1, \cdots, p_4)$, $(q_1, \cdots, q_4)$ for which (p_1, q_1) coincide at a fixed point O, there is obtained a group in ∞ to 1 isomorphism with the plane Cremona group which I⁽³⁰⁾ have called the *dilation* of the plane Cremona group. It is, in fact, considered as a group on the net of lines through O, indentical with the planar group. If the pairs (p_1, p_2), (q_1, q_2) coincide at (O_1, O_2), we have a dilation of the planar Jonquière group. Finally if the triads (p_1, p_2, p_3), (q_1, q_2, q_3) coincide at (O_1, O_2, O_3), we have the special case of the group of Young and Morgan for which the three lines form a triangle. For all of these cases the extensions to higher space are more or less mechanical.

6. *Fixed Points and Cyclic Sets.* The transformation in the plane of order n, C_n, has $n + 2$ fixed points defined by the vanishing of the matrix

$$\left\|\begin{matrix} x_0 & x_1 & x_2 \\ f_0 & f_1 & f_2 \end{matrix}\right\|$$

outside the F-points. It may, however, have a curve of fixed points accounting for some or all of the isolated fixed points. The order of C_n^2 is $m \leqq n^2$. As a rule $m > n$, so that C_n^2 has more fixed points than C_n. The extra fixed points arise from those pairs which are interchanged by C_n. Similarly, the fixed points of C_n^3 are either fixed points of C_n or points of cyclic triads of C_n. Proceeding in this way, the number of cyclic k-ads may be obtained, a number which must be modified if at any time a locus of ∞^1 cyclic k'-ads (k' a factor of k) should arise. In 1866 Reye⁽³¹⁾ discussed the four fixed points of the quadratic transformation and in 1880 S. Kantor⁽³²⁾ determined its cyclic triads. It must be remarked, however, that the fixed points, and in general the cyclic sets, play a minor rôle in the transformation as compared with the F-points.

If every point of the plane is a member of a cyclic k-ad (or for particular points a member of a cyclic k'-ad, where k' is a factor of k), the transformation is *periodic* of period k. The most interesting case is the *involutory* transformation. Bertini⁽³³⁾ has proved that any involution can be transformed into one of four types: (*a*) the harmonic homology; (*b*) the Jonquière involution of order n; (*c*) the Geiser involution of order 8; and (*d*) the Bertini involution of order 17. An involution interchanges a line and an n-ic curve which meets the line in $2k$ involutory pairs and $n-2k$ points on the curve of fixed points. Here k is the so-called *class* of the involution. Involutions of successive classes have been studied by Bertini(34), Martinetti(35), and Berzolari(36), and the types found by them have been reduced by Morgan(37) to the four types of Bertini. However, the class of an involution is a projective rather than a cremonian invariant number for the involution.

For the greater part of our knowledge of transformations of higher period and of finite Cremona groups, we are indebted to the genius of the great geometer S. Kantor. His work in this field is remarkable both for breadth of view and mastery of complicated detail. I shall not have space to outline his methods, which, as set forth in his crowned memoir(38) and his book(39), have been reviewed by Caporali(40) and by Kantor(41) himself. His results have been corrected in important but not essential particulars by Wiman(42). Kantor has considered also periodic transformations in space(43) and has derived the finite groups of regular transformations in S_3(44).

C. GEOMETRIC APPLICATIONS

7. *Canonical Forms.* The applications of Cremona transformations to geometry are more or less adventitious. Broadly speaking, transformations enter into geometry in one of three ways. First, the geometric properties of a given transformation may be studied and it is very commonly true that particular transformations define geometric figures of great interest. Instances of this are given in § 9. Secondly, the elements of a given geometric problem may be such that some or all of them serve to define a transformation whose known properties illuminate the given problem. Third, a given geometric problem may be transformed into a simpler form by a properly chosen transformation. The last is perhaps the most common use of transformations.

In birational geometry, a particular curve or surface is usually studied after it has been birationally transformed into a canonical or normal form. So also in problems which involve linear systems of curves or of surfaces, it is usually convenient first to transform the given system by a Cremona transformation into the simplest possible form before discussing its properties.

In connection with his discussion of involutions Bertini(33) has shown that a pencil or a *complete* net of rational curves (i.e., a net determined by base points alone) can be transformed into a pencil or net of lines; a pencil of elliptic curves into a pencil of order $3r$ with 9 r-fold points; a net of elliptic curves into a net of cubics on 7 points; and a web of genus 2 with 4 variable intersections into the web of sextics with 8 given nodes. Guccia(45) extends these results to linear systems of any dimension of rational and elliptic curves, and for the latter case finds that all can be transformed into either a linear system of elliptic cubics with $v = 0, 1, \cdots, 7$ simple base points; or into a linear system of quartics with two nodes; or into the pencil of order $3r$ with 9 r-fold points. Martinetti(46) considers linear systems of genus 2, and Yung(47) linear systems of any genus. These investigations are reviewed and confirmed by Ferretti(48), following Castelnuovo's revised treatment of the homoloidal net. S. Kantor(49) developed an equivalence theory for linear systems of rational, elliptic, and hyperelliptic curves on the hypothesis that the base points of the system of like multiplicity are not to be separated by the use of algebraic irrationalities, and thereby naturally obtained a larger number of types.

These canonical forms are very useful in connection with mapping problems. When we set as in (1), (3) 2

$$y_i = f_i(x_0, x_1, x_2) = f_i'(x_0', x_1', x_2'), \quad (i = 0, 1, \cdots, d),$$

where x, x' are co-points under a Cremona transformation, and thereby map the plane E_x upon a 2-way in S_d by means of the linear system of curves f_i, then the linear system f_i' determines precisely the same map. Evidently the essential peculiarities of the map will be more clearly presented if the mapping system is first reduced to a normal form.

If $d = 3$, these maps are the rational surfaces. By these methods Picard(50) shows that the only surfaces whose plane sections are rational are the Steiner quartic and the rational ruled surfaces, and Noether(51) determines the types of rational surfaces of order 4. Other writers along this line are Caporali, Segre, Del Pezzo and Castelnuovo. If $d = 2$, the map reduces to a plane E_y, and there is established a (1 to k) correspondence between E_y and E_x. Classifications of such correspondences for simpler values of k have been made by Sharpe and Snyder(52) and by others.

8. *Birational Transformations.* We may define the birational group of a curve or surface to be that aggregate of birational transformations which transforms the given curve or surface into the totality of curves or surfaces, respectively, which are of the same *class,* that is, birationally equivalent to the original curve or surface. Under this definition, any Cremona transformation belongs to the birational group. The converse is not true, however. For example, the rational plane sextic curve can be birationally transformed into a line, but its order cannot be reduced by plane Cremona transformation. If, however, a birational transformation under consideration, say from plane curve to plane curve, can be effected by a Cremona transformation, then the existence of the latter will necessarily throw much light on the former. For the behavior of the transformation off the curve conditions its behavior on the curve. This transition from birational to Cremona transformation is usually not unique. Thus given a cubic curve, $(\alpha x)^3 = 0$, the pairs of points x, x' for which $(\alpha x)(\alpha x')\alpha_i = 0$ $(i = 0, 1, 2)$ are corresponding pairs of a birational transformation of the hessian cubic into itself. These pairs x, x' satisfy a net of bilinear relations, and any one of the ∞^2 pencils of bilinear relations in this net determines an involutory quadratic transformation which effects the given correspondence on the hessian. This system of quadratic

involutions is of great help in studying the properties of the hessian as related to the correspondence upon it. More pretentious examples of this sort in connection with surfaces are found in papers of Snyder(53) and of Snyder and Sharpe(54).

9. *Geometric Figures attached to Cremona Transformations and Groups.* In this and the following section I wish merely to indicate some interesting geometric contacts of such particular Cremona transformations and groups.

The Geiser(55) involution G has copoints (x, x') which make up with 7 given points the 9 base points of a pencil of cubics. The locus of lines on which (x, x') coincide is a general quartic envelope for which the 7 points form an Aronhold system of double points.

The Bertini involution B has copoints (x, x') which are simple base points of a net of sextics with nodes at 8 given points. Its fixed points are the ninth nodes of such sextics. It isolates the 120 tritangent planes of a space sextic of genus 4 on a quadric cone(56).

The Geiser(55) involution in space G' has copoints x, x' which make up with 6 given points the 8 base points of a net of quadrics. Its locus of fixed points is the Weddle quartic surface.

The Kantor(57) involution in space K is defined by the fact that quartic surfaces with nodes at 7 given points and on a point x meet in another point x'. The ten nodes of a Cayley symmetroid quartic surface have the characteristic property that the involution K determined by any seven of the nodes has the other three nodes for fixed points.

The system of cubic curves on 5 points of the plane is unaltered by a Cremona G_{16} with quadratic elements. The plane is mapped by this system upon a 2-way of order 4 in S_4 which admits 16 collineations. Projected from a point not on it this 2-way becomes a quartic surface with a double conic(58); from a point on it a general cubic surface(59).

The extension of this group to space is a Cremona G_{32} which leaves a Weddle surface unaltered. Quadrics on the six nodes of the Weddle surface map it upon a Kummer quartic surface and the G_{32} becomes the collineation G_{16} of the Kummer surface(60).

The two groups just defined can be generalized to a $G_{2^{k+2}}$ in S_k determined by $k + 3$ points, but the further cases have not yet been studied. It is worth noting that in the cases for which k is odd the group will contain an involution like G' which is symmetrically related to the whole set of $k + 3$ points (see (23), p. 369, (29)).

The cross-ratio group of Moore(61) of order $(k + 3)!$ is determined by a base ($k + 2$ points) in S_k. It permutes the system of rational norm curves on the base with the $k + 2$ systems of lines on each of the base points. The form problem of this group consists in the determination of k independent cross-ratios of a binary $(k + 3)$-ic whose invariants are given.

It is clear from the above instances that Cremona transformations are intimately related to important geometric configurations. However the real utility of such transformations for geometric investigation can be realized only after closer study. (See, e.g., Conner, loc. cit. (57).)

D. Algebraic Applications

10. *Difficulties.* I have remarked in the introduction that there exists as yet for the Cremona group no distinctive geometry and no distinctive invariant theory. Much of the literature at hand deals with the transformations in a descriptive way, i.e., their projective properties are developed. On the other hand, writers continually use such properties as the invariance of the genus or the permanence of linear series on an algebraic curve under Cremona transformation; but these are properties which belong properly to the birational group rather than to its Cremona subgroup. Fano(62) remarks:

> "It may be a question whether there *are* properties of curves and surfaces not invariant under the birational group but yet invariant under the Cremona group. On this point no systematic investigations have yet been made."

As against this view, we observe that any rational curve can be birationally transformed into a straight line, but that not every rational curve can be so reduced by a Cremona transformation. There must in the nature of the case be *some* way of stating this latter possibility. What is desired is a language whose terms are invariants of the Cremona group but of no larger group. In projective geometry we speak of a *curve of the nth order;* in birational geometry of a *curve,* including under the term any one of the entire algebraic class; the corresponding term in cremonian geometry has as yet no well-defined content. Though the number of cremonian words is quite small they are not entirely lacking. Thus the term "linear system of curves" is cremonian, for a Cremona transformation transforms such a system into a similar system, whereas a birational transformation, being defined on only a single curve, cannot affect a system. Again the theorem:—"The jacobian of a net of curves is a covariant of the net"—belongs not merely to projective geometry for which it is commonly proved, but rather to cremonian geometry. For if the net N is transformed by any Cremona transformation into a net N', the jacobian J_N of N, the locus of additional nodes of curves of the net N, is necessarily transformed into the jacobian $J_{N'}$ of the net N'. The very meagreness of the cremonian vocabulary ensures that every additional term introduced into it will lead to a considerable enrichment of the geometry. I trust that this statement will be confirmed when the term *congruence of point sets under Cremona transformation* is discussed later. One should not overlook, however, the satisfactory state of two important problems in the plane; namely, the determination of reduced types for linear systems and for finite subgroups.

11. *Two Fields that admit an Invariant Theory.* Two things must be present before an invariant theory is possible: (1) a group of operations and (2) a field of objects permuted by the elements of the group. These are the bare essentials. In order that such a theory shall be fruitful it is also necessary

that the group shall be reasonably significant and that the field of objects shall be reasonably uniform so that an object and its transform may have in common a sufficient body of properties to form a reasonable basis of comparison. The Cremona group is significant enough, but it is hard to find fairly permanent qualities in any field of objects. Thus the order and the singularities of a single curve as well as a contact of two curves may all be altered by a properly chosen transformation. To be sure the totality of linear systems of curves has the permanence required but the objects of this field are as vague as they are general. Furthermore it is desirable to have a second field of objects so that the theory may present something of the nature of duality.

By the introduction of the class of sets of n points in the plane, sets P_n^2, congruent to each other under Cremona transformation, we obtain a new field of objects permuted under the Cremona group which is in a sense dual to the field of linear systems. Moreover the objects of this field are so concrete and so easily visualized that some success in the development of their invariantive properties may well be expected. The definition of this class is as follows: The planar set P_n^2 of points $(p_1, p_2, \cdots, p_n)$ is congruent to the planar set Q_n^2 of points $(q_1, q_2, \cdots, q_n)$ if there exists a Cremona transformation C_m with $\rho \leqq n$ F-points in the set P_n^2 and ρ inverse F-points in the set Q_n^2 for which the remaining $n - \rho$ points in either set form $n - \rho$ copairs of the transformation C_m.

The invariance of this class under Cremona transformation is an immediate consequence of the definition. For if P_n^2 is congruent to Q_n^2 under C_m and Q_n^2 is congruent to R_n^2 under $C_{m'}'$, then P_n^2 is congruent to R_n^2 under the product $C_m \cdot C_{m'}'$. Nevertheless, certain points relating to the definition should be emphasized. We note first that, P_n^2 being given, not all Cremona transformations are eligible for the formation of the class of congruent sets, but only those with not more than n F-points, and of these only those whose F-points are found in the set P_n^2. This is, to be sure, a limitation, but in applying the operations of any group, we are entitled to prescribe any rules which do not contradict properties inherent in the elements. Moreover the limitation is not serious, for n may be chosen in advance sufficiently large to include any desired point set or any desired transformation.

Again we observe that if, in particular, C_m is a projectivity C_1, the projectively equivalent sets P_n^2, Q_n^2 satisfy the definition of congruence. This of course must be expected since the projective group is a subgroup of the Cremona group. It raises the question, however, as to whether for other Cremona transformations, congruence may not mean merely projectivity. I have shown ((23), p. 353) that this can occur in only four cases, namely, for sets of 5 points under C_2 and C_3; for sets of 7 points under the Geiser C_8; and for sets of 8 points under the Bertini C_{17}. For each of these cases, the introduction of a single additional copair destroys the projectivity.

Finally we observe that congruent point sets are not birationally equivalent. Consider for example an elliptic cubic K^3 on a set P_6^2. A quadratic transformation, A_{123}, with F-points (p_1, p_2, p_3), (q_1, q_2, q_3) and copairs (p_i, q_i), $(i > 3)$, transforms it into an elliptic cubic $K^{3'}$ on the congruent set Q_6^2. In the birational transformation thus effected from K^3 to $K^{3'}$, the set P_6^2 is not equivalent to the congruent set Q_6^2 but rather to that set on $K^{3'}$ which consists of the three points where the sides of the F-triangle (q_1, q_2, q_3) meet $K^{3'}$ again and of the three ordinary points (q_4, q_5, q_6).

Hence in the congruence of point sets we have a notion peculiar to cremonian geometry. It is clear also that the conditions for congruence imply when $n = \rho$ the conditions on the two sets of F-points that the transformation may exist, and when $n = \rho + 1$ the further conditions for the construction of the transformation ((23), § 2). This notion originated in my own mind through the contrast presented by the two algebraic problems of the next section.

12. *Form Problems of Cremona Groups.* A proper conic with three distinct points marked on it is projectively equivalent

to any proper conic with three distinct points on it. We select then such a conic N, and establish on it a parameter system t such that three given points on N have parameters $t = 0, 1, \infty$. Binary quintics, $(\alpha t)^5 = 0$, then determine on N sets of five points. We project four points of such a set into the vertices of the reference triangle and the unit point, and the fifth point takes the position x (x_0, x_1, x_2). Corresponding to the various orders in which this projection can be made, we get 120 points x, a conjugate set under Moore's G_{120}. Fundamental regions for this group have been given by Slaught⁽⁶³⁾. Thus an ordered binary quintic determines a point x and conversely a point x determines a binary quintic to within projective modification. An invariant of the quintic is symmetric in the roots. Hence the locus of points x for which an invariant of the quintic vanishes is necessarily an invariant curve of G_{120}. The converse of this is also true. The quintic has invariants I_4, I_8, I_{12} of the degrees indicated, and to these there correspond invariant curves of G_{120}, J_6, J_{12}, J_{18} of the orders indicated with multiple points of orders 2, 4, 6, respectively, at the four base points. If the values of the absolute invariants I_8/I_4^2, I_{12}/I_4^3 of the quintic are given, the point x must be on the curves $J_{12}/J_6^2 = I_8/I_4^2$, $J_{18}/J_6^3 = I_{12}/I_4^3$. These meet in $12 \times 18 - 4 \times 4 \times 6 = 120$ points x outside the base points, a conjugate set under G_{120}. The form-problem of G_{120} consists in the determination of one of these points x when the values of the absolute invariants are given. One point is sufficient, since the others are obtained from one by effecting the known operations of G_{120}. If this one point is obtained, the given quintic can be solved by rational processes. For x determines the roots t' of a binary quintic which is projective to the given quintic with roots t. The determination of a linear transformation which sends the known quintic into the required quintic is a long-known exercise in binary forms. Thus the solution of the quintic is reduced to the solution of the form-problem of G_{120}. I shall now show that the form-problem of G_{120} can be solved in terms of Klein's⁽⁶⁴⁾ problem of the A's.

To invariants of the quintic there correspond invariant curves of G_{120}; but to irrational invariants of the quintic there correspond members of a linear system of curves invariant under G_{120}. For example, to the irrational invariant $\overline{12}\cdot\overline{23}\cdot\overline{34}\cdot\overline{45}\cdot\overline{51}$ ($ij = t_i - t_j$) of weight 5 and degree 2 in each root there corresponds a cubic curve on the four base points. The cubics determined by the 12 conjugates of this irrational invariant all lie in the linear system $\lambda_1 D_1 + \cdots + \lambda_6 D_6$ of cubics on the four base points, the simplest linear system unaltered by G_{120}. Adjoin now the square root of the discriminant Δ of the quintic, an invariant of degree 8. The group reduces to the even G_{60}, the invariants to I_4, $\sqrt{\Delta}$, I_{12}, the invariant curve which corresponds to $\sqrt{\Delta}$ being the six lines on the four base points. Also the above linear system of cubics separates into two nets

$$\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2, \qquad \mu_0 B_0 + \mu_1 B_1 + \mu_2 B_2,$$

each net being invariant under the Cremona G_{60}. Indeed under G_{60} the members of the first net experience the linear transformations of Klein's group of the A's, a ternary linear g_{60}; and the second net, Klein's group of the B's which arises from that of the A's by replacing ϵ by ϵ^2 ($\epsilon = e^{2\pi i/5}$). The invariants K_2, K_6, K_{10} of g_{60} of orders 2, 6, 10 must be calculated in terms of the known invariants of the quintic, and the same must be done for three invariant forms L_3, L_5, L_7, linear in the B's, and of the degrees indicated in the A's. The form-problem of the A's consists of the determination of A_0, A_1, A_2 when the numerical values of K_2, K_6, K_{10} are given. When this has been done, we find from the known values of the forms L_3, L_5, L_7 and the A_0, A_1, A_2 the values of B_0, B_1, B_2. Knowing A_0, A_1, A_2 and B_0, B_1, B_2, the coordinates of x are easily obtained. Thus the form-problem of the Cremona G_{120} is solved in terms of that of the A's. I have given(65) a solution of the A-problem in terms of the binary ikosahedral form-problem, and have carried out explicitly all the calculations necessary for this as well as for the steps outlined above. Finally the binary form-problem is solved in terms of elliptic modular functions. (See Klein, loc. cit. (63), p. 132.) The chief rôle of the cross-ratio group in this procedure for the solution of the quintic is to furnish by all odds the most natural reduction to the problem of the A's. It eliminates at one stroke three parameters from the five involved in the given quintic.

I have also discussed(66) the connection of the cross-ratio group with the solution of the sextic and found that it correlated well with the current theory.

These cross-ratio groups are associated respectively with 5 points in the plane, 6 points in space, etc. It is of course a natural question as to whether there is an analogous theory for further sets of points. The case of 6 points in the plane is next in order. When we transform as before four of the six to the reference triangle and unit point, the other two become x_0, x_1, x_2 and y_0, y_1, y_2. In each of these there is an unessential factor of proportionality. One of these factors is removed by requiring that the last coordinate of each shall be the same, leaving one factor of proportionality common to the two points. Thus P_6^2 has acquired the *canonical form*

$$\begin{array}{ccc} 1, 0, 0 & \quad & 1,\ 1, 1 \\ 0, 1, 0 & & x_0, x_1, u \\ 0, 0, 1 & & y_0, y_1, u \end{array}.$$

It is represented by a point P in a linear space Σ_4 with coordinates x_0, x_1, y_0, y_1, u. Obviously all projectively equivalent sets P_6^2 are represented by the same point P in Σ_4; and, conversely, any point P in Σ_4 represents a class of projectively equivalent sets P_6^2. The ratios of the coordinates of P are absolute projective invariants of the set. If, however, a given set P_6^2 is transformed into the base points, and a fifth and sixth point in a different order, a new representative point P' in Σ_4 is obtained. The coordinates of P' are rational in those of P, and thereby for all possible permutations of P_6^2 we find in Σ_4 a set of 6! points P conjugate to each other under a Cremona $G_{6!}$. This is the extension to the larger set of the Moore cross-ratio group. It puts into evidence the projective invariants of the set. These when symmetric are the invariants of the group. I have developed(67) the theory of this group for the set P_n^k of n points in S_k and have given a complete system of invariants for P_6^2. Complete systems for further sets would be very useful but they are hard to find.

If to P_6^2 in canonical form we add a point z_0, z_1, u to form P_7^2, the same canonical coordinates appear in the set P_7^3 in S_3, namely:

$$\begin{array}{llll}
x_0, & y_0, & z_0, & u \qquad 0, 0, 0, 1 \\
x_1, & y_1, & z_1, & u \qquad 0, 0, 1, 0 \quad . \\
1, & 1, & 1, & 1 \qquad 0, 1, 0, 0 \\
 & & & \qquad\;\; 1, 0, 0, 0
\end{array}$$

These two ordered sets, the one is S_2 and the other in S_3, are called *associated*. In the general case a set P_n^k is associated with a set P_n^{n-k-2}. Such associated sets are in remarkably intimate relation both in a geometric and algebraic way. For example, they have the same invariants under $G_{n!}$.

If P_6^2 yielded no more than this $G_{6!}$ it would have no advantage over P_6^3 with its cross-ratio $G_{6!}$. Let us see then whether we cannot obtain a little more from P_6^2. To the geometer 6 points in the plane always suggest the cubic surface Γ^3 mapped from the plane by cubic curves on the six points. The directions about each of the 6 points map into 6 skew lines—a line-six—on the surface. The six conics on 5 of the 6 points map into a similar line-six, cutting the first line-six five lines at a time. The two line-sixes form a double-six. The 15 lines on two of the six points map into the remaining 15 lines of the surface. In other words, P_6^2 defines a cubic surface with an isolated line-six. It is known, however, that there are 72 such line-sixes on the surface. They arise as follows. The quadratic transformation A_{123} sends cubics C_i on P_6^2 into cubics C_i' on Q_6^2. The mapping equations are

$$y_i = C_i(x) = C_i'(x'), \qquad (i = 0, 1, 2, 3).$$

Thus the point y on the surface determined by x or x' does not change but the line-six determined by the points of Q_6^2 is that determined at P_6^2 by the sides of the triangle p_1, p_2, p_3 and the points p_4, p_5, p_6. Hence the passage from P_6^2 to the congruent set Q_6^2 corresponds on the surface to the passage from one line-six to another. Thus there must be 72 projectively distinct sets P_6^2 congruent under ternary Cremona transformation to a given set, and these in all possible orders correspond in Σ_4 to $72 \times 6! = 51{,}840$ points P conjugate under a Cremona G_{51840} isomorphic with the group of the lines on a cubic surface and furthermore in immediate algebraic relation to the surface itself. I have called this the *extended group* $G_{6,2}$ of P_6^2 ((23), §§ 3, 4). Its invariants are the invariants of the cubic surface. Its form-problem can be solved in terms of that of Burkhardt's linear group of the Y's which in turn can be solved in terms of hyperelliptic modular functions of genus 2 and rank 3. The group $G_{6,2}$ plays the same rôle in the determination of the 27 lines on a cubic surface as does the cross-ratio group in the determination of the roots of a quintic(68). Indeed these two problems, which from the Galois point of view present at most a superficial resemblance, can be developed in a series of practically identical stages both in the algebraic and modular function fields. (See (68), p. 372.) This analogy is so perfect as to justify in itself the methods here employed.

The extended group, based as it is on the canonical form of the point set, and on congruence of point sets, can be generalized immediately to the set P_n^k. The associated sets P_n^k and P_n^{n-k-2} have the same extended group. The study of the structure of these extended groups is much simplified by the fact that they are isomorphic to the corresponding arithmetic group of section 4, a linear group $g_{n,k}$ ((63), § 5) with integral coefficients, and eventually in a smaller number of variables than $G_{n,k}$. Except for the sets P_6^2, P_7^2, P_8^2, P_7^3, P_8^4, the number of congruent sets is infinite, and $G_{n,k}$ is infinite and discontinuous. The theory then passes from the algebraic field into that of automorphic functions.

13. *Invariants of a Set* P_n^k. We define an invariant of a set P_n^k under regular Cremona transformation to be an algebraic form in the coordinates of each point of the set which is symmetric in the n points and which is reproduced to within a factor when formed for a congruent set Q_n^k. It is therefore an invariant under $G_{n,k}$. Since projective sets are

congruent also, the invariant must be a sum of terms each of which is a product of determinants formed from the coordinates. We say that the product $\overline{123}\cdot\overline{456}$ formed for P_6^2 is an irrational invariant, since each point of P_6^2 occurs linearly but the symmetry is not present. There are 10 such products and they lie in a linear system of irrational invariants of dimension 4, i.e., all can be expressed linearly with numerical coefficients in terms of 5 of them. In $\Sigma_1^{10}(\overline{123}\cdot\overline{456})^2$ we have an invariant of P_6^2 under $G_{6!}$ but not under the extended group $G_{6,2}$. (See (67), §§ 3, 4, 5.)

To make this extension consider the meaning of the determinant factors $\overline{ijk}$ in the products from which invariants are formed. The factor $\overline{ijk}$ vanishes when p_i, p_j, p_k are on a line. This means that the cubic surface mapped from P_6^2 has a node. But it also has a node when two of the points coincide in a definite direction, or when the six points are on a conic. Under the quadratic transformation A_{123}, the condition $\overline{124}=0$ on P_6^2 becomes for Q_6^2 the condition that q_3, q_4 coincide, and $\overline{456}=0$ becomes the condition that the six points q are on a conic. We have then 36 discriminant conditions for the set P_6^2, namely, 15 of the type δ_{ij}, which vanishes if p_i, p_j coincide; 20 of the type δ_{ijk}, which vanishes if p_i, p_j, p_k are on a line; and one of the type δ, which vanishes if all six are on a conic. These discriminant conditions are permuted under $G_{6,2}$ like the 36 double-sixes on the cubic surface. Consider now the product $d_2\,\overline{123}\cdot\overline{456}$, where $d_2=0$ is the condition of degree two in each point that the six are on a conic. The product is of degree 3 in each point and vanishes once for each coincidence but once more for the coincidences δ_{12}, δ_{13}, δ_{23}, δ_{45}, δ_{46}, δ_{56}, whence it represents a product of the 9 discriminant conditions δ, δ_{123}, δ_{456}; δ_{12}, δ_{13}, δ_{23}; δ_{45}, δ_{46}, δ_{56}. Such a product is one of 40 similar products conjugate under G_{51840}.

They correspond on the surface to the 40 so-called *complexes*. By forming such combinations of these products as are invariant under G_{51840} we obtain the invariants of P_6^2 under the extended group $G_{6,2}$. They are in effect the invariants of the cubic surface and their complete system is known. (See (68), §§ 1, 3.)

Similarly the invariants under $G_{7,2}$ of P_7^2 are the invariants of the quartic envelope determined as in § 9 by the seven points. Their complete system has not been found, but individual members can be obtained by symmetrizing a system of irrational invariants derived from the type

$$\overline{531}\cdot\overline{461}\cdot\overline{342}\cdot\overline{562}\cdot\overline{547}\cdot\overline{217}\cdot\overline{367}.$$

This type is of degree 3 for each point, vanishes once for each coincidence δ_{ij} and in addition vanishes for the seven discriminant factors $\delta_{531}, \cdots, \delta_{367}$. Here again the 63 discriminant factors for the set P_7^2 are the factors of the discriminant of the quartic envelope.

For larger sets beyond P_8^2 in the plane the number of discriminant conditions is infinite and the symmetrizing process under $G_{n,k}$ can no longer be used. Nevertheless in some of these infinite cases algebraic invariants of $G_{n,k}$ do exist. For example, on P_9^2 there is a unique cubic curve, and the invariants S, T of this cubic are invariants of P_9^2 under regular Cremona transformation. For P_{10}^2 the condition that the 10 points be on a cubic curve is a similar invariant. Other instances are for P_9^3 the condition that the 9 points lie on a nodal quadric, and for P_9^4 the condition that the 9 points lie on a rational quintic curve. These and other examples are all suggested by geometric considerations. On the algebraic side, there is no evidence that the invariants which exist for the subgroup $G_{n!}$ can be so combined as to form invariants of $G_{n,k}$ when $G_{n,k}$ is infinite. If the algebraic invariants fail, automorphic functions of the coordinates may replace them.

E. Applications to Modular Functions

14. *Modular Functions of Genus Three.* Our notion of a *modular function* attached to a general algebraic curve is dependent of course upon our definition of the *moduli* of the curve. These may be defined from the algebraic point of view as a set of $3p - 3$ independent projective absolute invariants of the normal curve C^{2p-2} in S_{p-1} upon which the

given curve is mapped by means of its canonical adjoints. They may also be defined from the transcendental point of view as the constants ω_{ij} $(i, j = 1, \cdots, p)$ which occur in the periods of the normalized integrals of the first kind attached to the curve. In the latter case, the $p(p+1)/2$ quantities $\omega_{ij} = \omega_{ji}$ are connected by certain relations when $p \geqq 4$ (thus far known only when $p = 4$) so that only $3p - 3$ are independent. In the first case one would naturally define a modular function to be an algebraic function of the algebraic moduli; in the second case an analytic function of the ω_{ij}. But to avoid the vagueness consequent to too great generality, $M(\omega_{ij})$ is restricted to be first a uniform function of the ω_{ij}, and second to be invariant under such integral transformations of the periods as form a subgroup of finite index under the totality of such transformations. It has thereby under the total group a finite number of conjugate functions and is therefore an algebraic modular function. Similarly only such algebraic moduli and such algebraic functions of these moduli are admitted as are uniform functions of ω_{ij}. Since the number of their conjugates is finite, each must be unaltered by a subgroup of finite index of the type mentioned.

The normal curve of genus 3 is the plane quartic which I take, for the moment, as an envelope of class 4. It has 288 Aronhold sets of 7 double points, sets P_7^2. If such a set be taken in canonical form, its representative point P in Σ_6 has coordinates x_0, x_1, y_0, y_1, z_0, z_1, u. The ratios of these 7 coordinates are algebraic moduli of the curve with precisely $7!288$ conjugate value systems which are conjugate also under the extended group $G_{7,2}$. All of the numerous relations among the 28 double points of C^4 are consequences of the statement that the 288 Aronhold sets P_7^2 are congruent under Cremona transformation. J. R. Conner(57) has derived some of these transformations from a different point of view.

If C^4, now a point curve, be mapped by cubics on the six contacts of a contact cubic, it becomes a sextic curve in space, the locus of nodes of quadrics of a net on 8 base points and the plane sections of this sextic furnish the contacts of a system of contact cubics. In general a set P_8^3 is congruent to infinitely many projectively distinct sets; but if P_8^3 is the set of 8 base points of a net of quadrics, I have shown ((23), p. 377 (45)) that it is congruent to only 36 such sets, each corresponding to one of the 36 systems of contact cubics. In each set P_8^3 there are 8 sets P_7^3 and thus there arise 8 × 36 sets P_7^3 which are the associated sets of the 288 Aronhold sets P_7^2. The relations among these 36 sets P_8^3 as well as the conditions on a particular set are all implied by their congruence properties.

The irrational invariant of P_7^2 found in § 13 is an algebraic modular function. It appears ((23), § 8) that it is one of 135 conjugates such that each one of the 135 can be linearly expressed in terms of 15 which themselves are linearly independent. Since these 15 are rational functions of only 6 moduli, they are related further, and it turns out that they satisfy a system of 63 conjugate cubic relations. It may be shown that these linear and cubic relations are sufficient to define the system of irrational invariants as functions of the 6 moduli. I shall now show how to obtain expressions for these 15 modular functions as uniform functions of ω_{ij}. Since we can express the six moduli rationally in terms of the 15 functions, we shall thereby have expressions for the algebraic moduli as uniform functions of the transcendental moduli.

Following a suggestion of Klein, Wirtinger(69) has generalized the Kummer surface defined by the theta functions of genus 2 into a p-way M_p in S_{2^p-1} of order $p!\ 2^{p-1}$ defined by the theta functions of genus p. For $p = 3$, this is a 3-way M_3^{24} of order 24 in S_7. The definition is as follows. The squares of the 64 odd and even thetas for $p = 3$ are even functions of the second order and zero characteristic so that only 8 are linearly independent. Using such a set of 8 as homogeneous coordinates of a point in S_7 then, as the variables $u = u_1, u_2, u_3$ change, the point in S_7 runs over the M_3^{24}. A point of M_3^{24} is thus determined by $\pm u$; in particular the 64 half-periods (including the zero half-period) determine 64 four-fold points on M_3^{24}. This M_3^{24} is transformed into itself by a collineation G_{64} whose elements are defined parametrically by $u' = u + \frac{1}{2}P$. The 8 theta functions are connected by a system of 70 quartic relations and Wirtinger(70) has proved that all other relations among them are a consequence of these; in other words, M_3^{24} is the complete intersection of 70 quartic spreads in S_7. But, contrary to the case when $p = 2$, there are here 8 cubic relations on the functions, so that 64 of the 70 quartic relations are merely a cubic relation multiplied by one of the 8 functions. I wish to establish these cubic relations.

It is not difficult to find for coordinates 8 combinations of the theta squares, say X_{ijk} of weights $i, j, k = 0, 1$ with the following properties. On adding one half-period the X_{ijk} with odd first weight i change sign (to within a factor common to all), on adding a second half-period those with odd second weight change sign, and similarly for the third. Thus, there arises a collineation G_8 of changes of sign. On adding another half-period the first weights $i = 0, 1$ interchange; similarly for the second, and for the third. Thus there arises a collineation G_8' of permutations of the X which combines with G_8 to yield G_{64}. Assuming now a general cubic relation, we obtain from it, by using G_8, 8 relations, and combine these to obtain a simple relation with only 15 undetermined coefficients $a_1, \cdots, a_{15}$. To this one we apply G_8', and get the 8 distinct cubic relations. The left members of these appear at once as the 8 first derivatives of a quartic relation with coefficients $a_1, \cdots, a_{15}$. Hence *the Wirtinger M_3^{24} is the manifold of double points on a unique quartic spread F^4 in S_7*. To determine the coefficients $a_1, \cdots, a_{15}$ of F^4 we observe that one of the 63 involutions in G_{64} has two fixed S_3' s, each of which cuts M_3^{24} in 16 points determined by specific quarter-periods, and permuted in the S_3 by a G_{16}. Hence this S_3 cuts F^4 in a Kummer surface whose 16 nodes are determined by the quarter-periods, and whose coefficients are linear in $a_1, \cdots, a_{15}$. But the coefficients of a Kummer surface satisfy a cubic relation(71), and thus we find that $a_1, \cdots, a_{15}$ satisfy 63 cubic relations and therefore are proportional to the 15 linearly independent irrational invariants of P_7^2. Moreover the expressions for the coefficients of the Kummer surface in terms of the coordinates of a node are known(71), and, since here the coordinates of a node are the values of X_{ij} for a definite quarter-period, the values of the coefficients $a_1, \cdots, a_{15}$ as uniform functions of ω_{ij} are determined. In turn the expressions for the irrational invariants of P_7^2 as modular functions are obtained.

We have already noted that when P_8^3 is a set of 8 base points of a net of quadrics, the number of sets congruent to it is not infinite as it is for the general set P_8^3. This phenomenon occurs in other cases. Indeed if $E^4(u)$ is the unique elliptic quartic on P_8^3 with elliptic parameter u such that $u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} \equiv 0 \pmod{\omega_1, \omega_2}$ is the coplanar condition, then any set P_8^3 for which $u_1 + \cdots + u_8 \equiv \omega/r$ (r an integer) has this property. Also in the plane any set P_9^2 which has a similar sum on the unique elliptic cubic through the points has the property. This includes for $r = 1$ the 9 base points of a pencil of cubics and for $r = 2$ the 9 nodes of an elliptic plane sextic. The same peculiarity appears for the ten nodes of a rational plane sextic, and for the ten nodes of a Cayley symmetroid. The reason is that the theorem which states that congruent sets are not projective is not necessarily valid for special point sets. Indeed the theorem was proved ((23), p. 355 (7)) on the hypothesis that the set P_n^k was special to the extent that its n points were taken on an elliptic norm-curve, but the assumption then was made that their elliptic parameters satisfied no linear relation with integral coefficients, an assumption not fulfilled in the cases mentioned above.

15. *Modular Functions of Genus Four.* Consider first the set P_{10}^2 of nodes $p_1, \cdots, p_{10}$ of the rational sextic. The Bertini involution B with F-points at $p_1, \cdots, p_8$ has fixed points p_9, p_{10} whence P_{10}^2 is congruent to itself under B. Moreover under the transform of B by any quadratic transformation, such as A_{ijk}, P_{10}^2 is congruent to itself. Hence P_{10}^2 is self-congruent under the $\binom{10}{8}$ involutions B with F-points at P_{10}^2 and under the conjugates of these involutions

in $g_{10,\,2}$. The element of the arithmetic group $g_{10,\,2}$ which corresponds to B has coefficients

$$\begin{array}{ccccccc}
17 & -6 & -6 & \cdots & -6 & 0 & 0 \\
6 & -3 & -2 & \cdots & -2 & 0 & 0 \\
6 & -2 & -3 & \cdots & -2 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot \\
6 & -2 & -2 & \cdots & -3 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}$$

and therefore reduced modulo 2 is congruent to the identity. Any transform of B is similarly reducible to the identity. It may be shown(30) that B and its conjugates generate the subgroup $g_{10,\,2}{}^{(2)}$ of elements of $g_{10,\,2}$ which are congruent to the identity modulo 2. Moreover $g_{10,\,2}{}^{(2)}$ is an invariant subgroup of $g_{10,\,2}$ of index $10!\,2^{13}\cdot 31\cdot 51$, whence this is the number of projectively distinct ordered P_{10}^{2}'s congruent to P_{10}^{2}; or, disregarding the ordering, the number of projectively distinct types of rational sextics which can be derived from a given one is precisely $2^{13}\cdot 31\cdot 51$. Two years earlier than the date (1919) of this result, Miss Hilda Hudson(72) had proved that if a discriminant condition vanished for the sextic, then the sextic could be transformed into one of only 5 kinds, a fact related to the more precise one above.

The factor group of $g_{10,\,2}{}^{(2)}$ under $g_{10,\,2}$ is isomorphic with that subgroup of the odd and even thetas for $p = 5$ which has an invariant even characteristic (loc. cit. (30), p. 248 (9)). In other words the projectively distinct types of congruent sextics are permuted under Cremona transformation according to a theta modular group of the type mentioned.

The set P_{10}^{3} of ten nodes of a symmetroid has properties much like the set P_{10}^{2} of nodes of a rational sextic. In this case, however, the symmetroid is unaltered not merely by the $g_{10,\,3}{}^{(2)}$ but by other transformations so that only $2^{8}\cdot 51$ projectively distinct symmetroids can be obtained from a given one by regular Cremona transformation and these types are permuted according to the entire theta modular group for $p = 4$. The dependence of these two configurations of nodes upon theta modular functions thus foreshadowed by the occurrence of these factor groups is confirmed by a result obtained by Schottky(73), who proved that by combining theta modular functions of genus four, the coordinates of a set of ten points are obtained which have a characteristic property of the nodes of the symmetroid. His method is not adequate, however, to solve the inverse problem of finding the curve or curves of genus 4 thus attached to a symmetroid. Moreover the rational sextic and the symmetroid mutually determine each other in the following fashion. The plane rational sextic is *conjugate* to a rational sextic in space, the line sections of the one and the plane sections of the other being apolar binary forms. The symmetroid is the quartic locus of planes which cut the space sextic in *catalectic* binary sextics (i.e., sextics reducible to a sum of three sixth powers), the nodal planes being those which cut the sextic in *cyclic* binary sextics (i.e., reducible to two fifth powers). Thus given the plane sextic there is one projectively definite symmetroid; given the symmetroid there are two plane sextics(74). It is this intimate connection between functions of genus 4 and genus 5 which I am attempting to unravel in articles which are appearing in abstract form in current numbers of the PROCEEDINGS OF THE NATIONAL ACADEMY(75).

I have remarked that if algebraic invariants of $G_{n,k}$ are lacking, automorphic functions may be available. I shall now indicate how perhaps such functions may be constructed for the set P_{10}^2 of nodes of a sextic. Since the sextic is congruent to only a finite number of projectively distinct types, it is unaltered by an infinite discontinuous Cremona group, namely the subgroup $\overline{G}$ of $G_{10,\,2}$ which is isomorphic to the subgroup $g_{10,\,2}^{(2)}$ of $g_{10,\,2}$. On the sextic with binary parameter τ, $\overline{G}$ determines a binary group of the form $\tau' = (a\tau + b)/(c\tau + d)$. Let C_i be the cubic curve in variables x on the 9 nodes other than p_i. Transforming the C_i by A_{jkl} we get either cubics C_i or elliptic quartics with a node at p_r and p_s and simple points at the other points p. Divide such a quartic by its canonical adjoint $\overline{p_r p_s x}$, a rational curve determined by points of P_{10}^2, and the quotient thus obtained is still homogeneous of degree 3 in x. If then we take all the elliptic transforms of a given C by $G_{10,2}$ and divide each by its canonical adjoint, we obtain an infinite sequence of such fractional terms. But these terms must be separated into 527 classes such that the terms in one class arise from one term of the class by those operations of $G_{10,2}$ which are in $\overline{G}$, since it is only for such operations that the set of nodes remains fixed. Divide each of the terms in a class by the product π of the ten quadratics q_i in τ determined by the nodal parameters. Consider now the value of such a term when for x we substitute the parametric coordinates of a point τ on the sextic. A term like C_i has for numerator a product of 9 quadratics q_i divided by π and becomes λ_i/q_i where λ_i is numerical; an elliptic quartic term has for numerator $\pi q_r q_s$, and for denominator $\pi q_r q_s q_{rs}$, where q_{rs} is the pair of further parameters of points where the canonical adjoint cuts the sextic outside the nodes, whence it is λ_{rs}/q_{rs}. In other words the terms all reduce to the inverses of those binary quadratics cut out on the sextic by the F-curves of all the transformations of $G_{10,2}$. For the terms in a particular class we determine the numerical factors λ so that these terms are conjugate under $\overline{G}$ and finally sum the $2j$th powers of the terms in each class. Thus for every value of j we have 527 series which formally are pseudo-invariant under $\overline{G}$ and satisfy for $j \geqq 2$ the convergence criterion of Poincaré. Assuming that not all of these vanish for all values of j, then the ratio of any two for the same j will be automorphic under $\overline{G}$ provided $\overline{G}$ is a discontinuous binary group of the type for which the series of Poincaré converge. On returning to the corresponding series in x, we should then have automorphic functions, homogeneous of degree 0 in x_0, x_1, x_2, which converge for a region of values x which includes at least certain regions on the sextic. I have some information concerning the group $\overline{G}$, but not enough to validate entirely the process outlined above. In passing from one sextic to a congruent sextic, the 527 series mentioned would be permuted like the remaining 527 even theta functions for $p = 5$ under the group which leaves one such function unaltered. Similar developments are possible for the symmetroid. If this procedure is valid, we should have functions of two or three variables automorphic under discontinuous groups of far more complex character than any hitherto considered.

16. *The Arithmetic Group. Dickson Groups.* It must be clear from the last section that the information concerning the Cremona group $G_{n,k}$ obtained from the isomorphic linear group $g_{n,k}$ is of considerable importance. This linear group with integer coefficients will have, for every value of the positive integer α, an invariant subgroup (necessarily of infinite order) which consists of those elements congruent to the identity modulo α. The factor group which in concrete form is merely the elements of $g_{n,k}$ reduced modulo α is necessarily finite. I have made(76) a study of this factor $g_{n,k}{}^{(\alpha)}$ for $\alpha = 2$. It appears that 16 cases are to be distinguished according as $n, k \equiv 0, 1, 2, 3 \pmod 4$. In all of these 16 cases, it turns out that this factor group either is itself a simple group, or contains an invariant abelian subgroup of low order whose factor group is simple. The simple groups thus obtained are of known types. Either they are the total group of the odd and even thetas for some value of p or the simple subgroup of this total group which leaves unaltered either an odd or an even function. In the notation of Dickson's *Linear Groups*, these are respectively the simple groups $A(2m, 2)$, $FH(2m, 2)$, and $SH(2m, 2)$.

The question as to the nature of these factor groups for larger values of α, and in particular for values $\alpha = p^n$ (p a prime), has not been touched. We remark of course that $g_{n,k}$ has an invariant linear form and an invariant quadratic form, so that the factor groups are either the linear groups of Dickson with a quadratic invariant or subgroups of them. In the latter case, it may be possible to obtain new series of groups; and even in the former case we should have some new light on the known group and a new application for it.

Investigations along these lines may advance the general problem of an explicit solution for the totality of types of Cremona transformation with a given number of F-points —a problem which amounts merely to asking for the coefficients r_i, s_j, α_{ik}, m of the general element of the arithmetic group $g_{n,k}$.

One would also naturally expect to find that the simpler transformations congruent to the identity modulo α would have interesting geometric properties comparable perhaps to those of the Bertini involution for $\alpha = 2$. It may well be that in this way other sets of points not less striking than the nodes of a rational sextic will be discovered.

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SHORTER NOTICES

Vorlesungen über algebraische Geometrie. Geometrie auf einer Kurve, Riemannsche Flächen, Abelsche Integrale. By Francesco Severi, translated by Eugen Löffler, with an introduction by A. Brill. Leipzig, B. G. Teubner, 1921. xvi + 408 pp.

There is no fault to find with Dr. Löffler's excellent work of translation, but one regrets to be deprived of the author's clear and limpid Tuscan. We must derive what consolation we may from the hope that what has been lost in artistic quality will be offset by reaching a wider public.

And this question of reaching a wider public is today a crucial one for algebraic geometry, and not for this branch of mathematics alone, be it said by the way. Grafted on function theory on the one hand and on pure geometry on the other, algebraic geometry has reached today, notably in Italy, a very considerable development. Few chapters of mathematics have greater artistic quality, few have evinced quicker growth and cry more loudly for systematic, careful, didactic exposition. This introductory work by one of the keenest and most elegant geometers of our day should be assured a royal welcome.

The Vorlesungen are composed of two distinct parts—a very complete exposition of the geometry on algebraic curves, indeed so complete that few questions of interest to geometers are left untouched, and a voluminous supplement. The first is a translation with some amplification of the author's 1908 lithographed lectures. Severi makes the whole theory rest squarely upon linear systems of curves in the plane—an excellent introduction to the more advanced parts of the theory. The author exhibits early his intense geometric bent, his desire to avoid analytic methods. Thus the reduction of singularities is treated without the Puiseux expansions. One misses somehow the finality which comes with these expansions, but the geometer will have his way. . . .

From linear systems of curves to linear series on a given curve is but a step. This is followed by operations on linear series, the extension of which to curves on algebraic surfaces has proved to be such a formidable tool in the hands of Italian geometers. Then come in order the extension to curves in any space and the introduction of the genus and the canonical series by the intrinsic method of Enriques (method of the Jacobian series). To the Riemann-Roche theorem we are led via the C. A. Scott proof of Nöther's fundamental theorem.

A particularly interesting chapter is the treatment of correspondences between two curves or between the points of a given curve. Severi's own work in this direction is of importance second only to that of Hurwitz. Of course his methods, largely geometric, could yield but little on singular correspondences, in the study of which transcendental methods are indispensable. However, the author's treatment has thrown much light on the subject and the connection which he established with an important class of surfaces must have had much to do with one of his main achievements —the discovery of the base for curves on an algebraic surface and its connection with Picard's transcendental theory.

The first part ends with the transcendental theory along the line of Picard's *Traité d'Analyse*, with some asides on real curves and questions of reducibility of abelian integrals, in which direction the author has also made important contributions.

We now come to the Anhang. The volume was ready in August, 1914. In the years that followed, the author, part of the time in active military service, was pursuing investigations on algebraic curves and these with some minor corrections form the subject-matter of the Anhang, which occupies the latter fourth of the work. In form it differs considerably from the rest and in ordinary times we should have expressed the wish to have it more completely merged with the rest, but no doubt this would have greatly delayed publication, and under the circumstances we feel thankful to have it as it is.

The Anhang is certainly the most interesting part of the work for the student of this subject, the one which he will want to read first. Greater preparation is required to read it than for the rest. The material, as interesting as it is new, consists primarily in a series of contributions to our very meager knowledge on families of algebraic curves with singularities assigned in type but not in position. A typical as well as interesting result is this: The manifold of all irreducible curves of given order and genus is itself irreducible. His results along this line lead the author to the first algebraic-geometric proof ever given of Riemann's existence theorem, and also to a new attack on the classification of curves in any space.

May this suffice to whet the reader's appetite and increase his desire to read Severi. He will not repent.

S. LEFSCHETZ.

Theoretical Mechanics. An Introductory Treatise on the Principles of Dynamics with Applications and Numerous Examples. By A. E. H. Love. Third Edition. Cambridge, University Press, 1921. xv + 310 pp.

The first edition of Love's Mechanics was published in 1897 and was reviewed in this BULLETIN in April, 1898. The second edition, published in 1906, differed from the first in that the material had been rearranged and rewritten, although the general content remained unaltered. The present edition is practically the same as the second, the only additions being a note on *The moment of the kinetic reaction of a particle about a moving axis*, a paragraph on *Force of simple harmonic type*, and a paragraph on *Effect of damping on forced oscillation.* The number of miscellaneous examples has been reduced, although the collection is still sufficiently large to satisfy the demands of the most ardent problems enthusiast.

As the first and second editions have been familiar to students of mechanics for so many years, no further comment seems necessary.

PETER FIELD.

Zur Theorie der Triaden, von Almar Naess. Kristiania, Grondahl & Sons, 1921. 136 pp.

This paper of 136 pages forms Nr. 6, Serie 1, of NORSK MATEMATISK

FORENINGS SKRIFTER, is clearly and fully written, and very interesting reading. An excellent outline of the properties of vectors and dyadics covers 32 pages. A triadic is then defined, as in the Gibbs-Wilson vector analysis, to be a sum of terms of the form **abc** where **a**, **b**, and **c** are vectors, and it is shown that the most general triadic can be written $\mathbf{i}\Psi_1 + \mathbf{j}\Psi_2 + \mathbf{k}\Psi_3$, where **i**, **j**, and **k** are rectangular unit vectors and Ψ_1, Ψ_2, and Ψ_3 are dyadics. It is this latter, semi-cartesian, form which is chiefly used in obtaining the results which follow.

It is a little surprising that, in a work otherwise so clearly permeated with the spirit of Gibbs, not more use is made of the first definition, from which, for example, the conjugate systems would follow much more compactly; for they are equivalent to permutations of the vectors in **abc**. If this be a fault, however, it is merely one of procedure, and in no sense implies any criticism of the spirit, purpose, or results of the work. A large number of identical relations are obtained, which appear to be useful, and these are frequently given in a final form free from the arbitrary units **i**, **j**, **k**, frequently in determinant form, often very elegantly conceived. In short, the semi-cartesian method is employed with great skill and leads naturally to the cubic matrix of the triadic. It would have appeared equally natural to introduce the cubic determinant. Dot products of triadics, and other products leading to tetradics, are also avoided.

An excellent feature is the continual application of the formulas to the differential operator (given its original name of Nabla). Students of vector analysis and related methods will await with interest further developments from the same pen.

F. L. HITCHCOCK.

Untersuchungen über das Endliche und das Unendliche. By C. Isenkrahe. Bonn, Marcus and Weber, 1920. Erstes Heft, viii + 224 pp.; zweites Heft, viii + 230 pp.; drittes Heft, ix + 245 pp. 16 mks. each.

These three divisions of the book under review are entitled:

1. Three detailed discussions of questions in the borderland between mathematics, natural science, and the theory of belief. 2. The teaching of St. Thomas as to the infinite, its application by Prof. Langenberg, and its relation to modern mathematics. 3. Letters between Sawicki and Isenkrahe on a problem in infinity, which is fundamental in the apologetic proof of entropy. The three divisions consist almost entirely of controversial matter in reply to various criticisms passed upon writings of Isenkrahe by Hartmann and others. Not much gain for the subjects under discussion is visible, much of the controversy seeming to relate to different uses of the terms. The discussions are of little interest mathematically. The author seems called upon to defend certain views for the sake of their possible religious significance. Recriminations of orthodoxy and unorthodoxy are found in places, and the argument waxes hot occasionally,—as one might expect.

J. B. SHAW.

The Copernicus of Antiquity (*Aristarchus of Samos*). By Sir Thomas Heath. London, Society for Promoting Christian Knowledge, and New York, The Macmillan Company, 1920. 59 pp.

It is always a matter for congratulation when a man of the highest scientific standing can be prevailed upon to give to the public a brief summary, in readable form, of results obtained in parts of his field. The little volume under review is such a contribution from the pen of the world's premier authority on Greek mathematics and mathematicians. Of the 56 pages of text, 37 are devoted to a short outline of the development of Greek astronomy before Aristarchus (that is, down to the 3d century B.C.), and it would be hard to find in so short a space an equally illuminating discussion of the subject.

The remainder of the text gives the 18 propositions of Aristarchus's only extant work, *On the sizes and distances of the sun and moon*, with a very brief indication of the methods of proof used. The most important of these propositions are the 7th, that the sun's distance from the earth is greater than 18, but less than 20, times that of the moon; and the 15th, that the diameter of the sun has to the diameter of the earth a ratio greater than 19 : 3 but less than 43 : 6. Although these results are wide of the mark, the methods used are entirely correct, and furnish a very interesting example of the way in which the Greeks used geometric inequalities to obtain numerical results which we now get from trigonometric tables.

Aristarchus's title, "The Copernicus of antiquity," rests upon the testimony of Archimedes and other ancient writers, who state explicitly that he taught the hypothesis of the double motion of the earth. It seems to have been the weight of Hipparchus's influence (2d century B.C.) which led to the rejection of this theory and the fixing of the geocentric theory in the position which it held unshaken for over 1500 years.

One who wishes fuller details as to Aristarchus and his work will of course turn to Sir Thomas Heath's *Aristarchus of Samos* (Oxford, 1913), which contains the complete Greek text and translation. For the general reader, however, the present work contains the gist of the matter (save for proofs). The one regrettable defect is the entire absence of figures. The two-page bibliography at the end is a useful addition.

R. B. McClenon

Éléments de la Théorie des Vecteurs et de la Géométrie Analytique. By Paul Appell. Paris, Payot et Cie, 1921. 147 pp.

This little volume in the Collection Payot gives in small space an introduction to the ideas of vectors. Coordinates are introduced as projections of a vector and vectors are represented throughout as a couple or a triple of coordinates. No attempt is made to bring in vector notation. It succeeds very well in setting forth the concepts of two and of three dimensions. No curve or surface is discussed at length. For those desiring a brief and very intelligible exposition of the elementary notions of vectors and analytic geometry it is recommended.

J. B. Shaw.

NOTES

The fifteenth regular meeting of the Southwestern Section of this Society will be held at the University of Kansas on Saturday, December 2, 1922, the first session opening at 10 A.M. There will be a smoker on the Friday evening preceding. On Saturday afternoon, Professor R. L. Moore, of the University of Texas, will give a paper by invitation of the program committee on *Continuous curves from the viewpoint of analysis situs.*

The annual meeting of this Society will be held at Harvard University, Cambridge, Massachusetts, on Wednesday and Thursday, December 27 and 28, 1922. Abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, not later than December 7. Professor E. H. Moore will give his retiring presidential address before the American Association for the Advancement of Science on the evening of December 26. Other features of the meeting will be the retiring address of President Bliss, and at least one joint session with the Mathematical Association of America and with Section A of the American Association for the Advancement of Science. A joint dinner of the three organizations will probably be held on the evening of December 28. It is presumed that the usual fare and one-half privilege will be granted for travel. Rooms in the Harvard dormitories are available for a limited number of men members of the Society, and in Radcliffe for a limited number of women, at the rate of $1.50 per day. Applications for these rooms should be in the hands of Professor O. D. Kellogg, 20 Craigie St., Cambridge, Mass., not later than December 5. Headquarters for registration will be at the Phillips Brooks House.

The Christmas meeting of the Chicago Section of this Society will be held at Northwestern University, Evanston, Illinois, on Friday and Saturday, December 29 and 30, 1922. Abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor Arnold Dresden, not later than December 7.

In May, 1921, the Petrograd Physico-Mathematical Society was founded at the University of Petrograd, on the occasion of the one hundredth anniversary of the birth of Tchebyscheff. Professor A. W. Wassilieff is president.

At Genoa a new society has been formed, known as Società Ligustica di Scienze e Lettere. Professor Gino Loria is secretary-general.

The University of Frankfurt and the Zurich Technical School have conferred honorary degrees on Professor David Hilbert, on the occasion of his sixtieth birthday.

The Technical School of Breslau has conferred the honorary degree of doctor of engineering on Professor A. Kneser, of the University of Breslau.

Professor J. Schur, of the University of Berlin, has been elected a member of the Berlin Akademie der Wissenschaften.

Professor A. N. Whitehead has been elected president of the Aristotelian Society for the coming session.

The Accademia dei Lincei has elected as corresponding members Professors P. Burgatti, L. Tonelli, and L. Lombardi; as foreign members Professors E. T. Whittaker of the University of Edinburgh and E. Landau of the University of Göttingen.

The University of Manchester has conferred the degree of doctor of science on Professor G. H. Hardy, of Oxford University.

The University of St. Andrews has conferred the honorary degree of doctor of laws on Sir P. B. S. Lang, emeritus professor of mathematics at that university.

Sir George Greenhill has been awarded a pension by the British government in recognition of his services to science and his ballistic work.

Professor A. E. Kennelly, of Harvard University, has received the Cross of the Legion of Honor from the French Republic for distinguished services as exchange professor of engineering.

Professor G. Loria of Genoa has been elected corresponding member of the Turin Academy of Sciences, and of the Czech Academy of Sciences and Arts.

On the occasion of the celebration of its seven hundredth anniversary the University of Padua conferred its honorary doctorate on Professors R. C. Archibald of Brown University, J. Lipka of the Massachusetts Institute of Technology, and V. Snyder of Cornell University.

The Indian Mathematical Society has elected to honorary membership Professors G. A. Miller of the University of Illinois and G. H. Hardy of Oxford University and Dr. G. T. Walker, Director-General of the Observatories in India.

In conversation with Professor Segre, members of the Society drew from him the reluctant confession that the ANNALI DI MATEMATICA is in straitened circumstances. The Italians have assisted to tide it over, during the last two years, and a number of banks have responded generously to its relief. No appeal was made to others, but more subscriptions would be welcome. With the approval by the Council of this Society at its Rochester meeting, Professor Snyder has undertaken to solicit subscriptions and cash contributions. The subscription is now forty lire (less than two dollars at present exchange) per volume. It appears quarterly, in numbers of about eighty pages. Its editors, Professors Bianchi, Jung, Pincherle, and Segre, insure that it will be continued as a mathematical periodical of the highest class. It is hoped that members of this Society will contribute generously. Contributions should be sent to Professor Virgil Snyder, 214 University Ave., Ithaca, New York.

Dr. K. Reidemeister has been appointed associate professor of mathematics at the University of Vienna.

Dr. P. Bernays has been promoted to an associate professorship at the University of Göttingen.

Dr. H. Falckenberg, of the University of Königsberg, has been appointed to an associate professorship at the University of Giessen.

At the University of Toronto Professor A. T. De Lury has been appointed Dean of the Faculty of Arts.

Dr. Norman Miller has been promoted to an associate professorship of mathematics at Queens University, Kingston.

At Dartmouth College, Assistant Professor R. D. Beetle has been promoted to a full professorship of mathematics, and Assistant Professor C. E. Wilder, of Northwestern University, has been appointed to an assistant professorship.

Assistant Professor J. H. M. Wedderburn has been promoted to an associate professorship of mathematics at Princeton University.

At Trinity College, Hartford, Mr. F. J. Burkett, of the University of Pittsburgh, has been appointed to an assistant professorship of mathematics, and Assistant Professor H. M. Dadourian has been promoted to an associate professorship of physics.

Assistant Professor C. C. Bramble has been promoted to an associate professorship of mathematics at the U. S. Naval Academy.

At Northwestern University, Professor F. E. Wood, of the Michigan Agricultural College, has been appointed to an assistant professorship of mathematics. Associate Professor Sinclair will be on leave of absence during the second term of 1922–23.

Professor C. G. Simpson, of the Carnegie Institute of Technology, has been appointed professor of mathematics at the Milwaukee College of Engineering.

Professor J. N. Donohue, of Notre Dame University, has been appointed professor of mathematics at Columbia University, Portland, Ore.

At the University of North Carolina, Assistant Professor E. T. Browne, of Trinity College, Hartford, has been appointed to an assistant professorship of mathematics, and Assistant Professor J. W. Lasley, Jr., has been promoted to an associate professorship.

At Clark University Assistant Professor C. E. Melville has been promoted to a full professorship.

At the University of Kansas Professor E. B. Stouffer has been appointed Acting Dean of the Graduate School.

On the return of pre-war conditions, the U. S. Naval Academy has reduced the number of its civilian officers of instruction in academic departments by about one third. From the department of mathematics, Assistant Professors R. P. Johnson and G. F. Alrich have accepted assistant professorships at the Carnegie Institute of Technology, and Mr. L. S. Johnston an assistant professorship at Pennsylvania State College.

At Indiana University, Associate Professor U. S. Hanna has been promoted to a full professorship, and Assistant Professor Cora B. Hennel to an associate professorship of mathematics.

Associate Professor Louis Brand, of the University of Cincinnati, has been promoted to a full professorship of mathematics.

Dr. N. R. Bryan has been appointed associate professor of mathematics at the University of Maine.

Mr. J. E. Davis, of the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Arkansas.

Mr. V. B. Hinsch, of the Missouri School of Mines, has been promoted to an assistant professorship of mathematics.

Assistant Professor E. D. Meacham, of the University of Oklahoma, has been promoted to an associate professorship of mathematics.

Assistant Professor W. L. Miser, of the Armour Institute of Technology, has been promoted to an associate professorship of mathematics.

At Washington University, Assistant Professor P. R. Rider has been promoted to an associate professorship of mathematics.

At the Georgia School of Technology, Associate Professor W. V. Skiles has been promoted to a full professorship, and Assistant Professors A. B. Morton and D. M. Smith to associate professorships of mathematics.

Assistant Professor R. B. Stone, of Purdue University, has been promoted to an associate professorship of mathematics.

At the University of Southern California, Dr. F. C. Touton, of the University of California, has been appointed associate professor of education.

Assistant Professor H. S. Uhler has been promoted to an associate professorship of physics at Yale University.

Assistant Professor Mary E. Wells, of Vassar College, has been promoted to an associate professorship of mathematics.

At Sophie Newcomb College, Tulane University, Dr. Anna M. Howe has been promoted to an assistant professorship and Miss Anna Nancy has been appointed assistant professor of mathematics.

Dr. Nina M. Alderton has been promoted to an assistant professorship of mathematics at Mills College.

At Miami University, Professor A. E. Young has resigned to accept a position with the Standard Oil Company at Pittsburgh. Associate Professor W. E. Anderson has been promoted to a full professorship, and succeeds him as head of the department of mathematics.

Miss Minna Schick, of the University of Minnesota, has been appointed associate professor of mathematics at the University of the Philippines.

Mr. B. D. Painter has been appointed assistant professor of mathematics at Hampden-Sidney College.

Mr. W. E. F. Appuhn, of the Stevens Institute of Technology, has been promoted to an assistant professorship.

Mr. H. W. Whetsell has been promoted to a professorship at Davis and Elkins College.

Miss Frances A. Atwater has been appointed head of the department of mathematics at Milwaukee-Downer College.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

ALLER (R. M.). Algoritmia. Principios fundamentales de la ciencias de los números. Coruña, 1918.

ALLIAUME (M.). See DE POORTER (A.).

ANTOINE (L.). Sur l'homéomorphie de deux figures et de leurs voisinages. (Thèse, Strasbourg.) Paris, Gauthier-Villars, 1921. 4to. 109 pp.

APPELL (P.). Eléments d'analyse mathématique à l'usage des candidats au certificat de mathématiques générales, des ingénieurs et des physiciens. 4e édition, entièrement refondue. Paris, Gauthier-Villars, 1921. 10 + 715 pp.

BIEBERBACH (L.). Differential- und Integralrechnung. Band I: Differentialrechnung. 2te, vermehrte und verbesserte Auflage. Leipzig, Teubner, 1922. 6 + 132 pp.

BRILL (A.). See SALMON (G.).

CIPOLLA (M.). Esercitazioni matematiche. Catania, 1921. 8vo. 52 pp.

DA CUNHA (P. J.). Reflexoes sobre a teoria dos conjuntos. Lisbon, Imprensa Nacional, 1922. 8vo. 64 pp.

DUPORT (—.). Le potentiel logarithmique. Paris, Gauthier-Villars, 1921. 8vo. 10 pp.

EULER (L.). Vollständige Anleitung zur Algebra. Neue Ausgabe. Leipzig, 1920.

FARKAS (A.). Memoir on some properties of indeterminate equations. Nagyenyed, Aiud, 1920. 9 pp.

FERROUX (G.). Mathématiques supérieures. Tome 1. Paris, 1921.

FIEDLER (W.). See SALMON (G.).

GOURSAT (E.). Leçons sur l'intégration des équations aux dérivées partielles du premier ordre. 2e édition. Paris, Hermann, 1921. 460 pp.

GRAY (A.) and MATHEWS (G. B.). A treatise on Bessel functions and their applications to physics. 2d edition, prepared by A. Gray and T. M. MacRobert. London and New York, Macmillan, 1922. 14 + 327 pp. $12.00

JUNKER (F.). Der grosse Fermatsche Satz. Vollständiger Beweis des Satzes auf elementar-mathematischer Grundlage. Stuttgart, Eugen Ulmer, 1921.

KOMMERELL (K.). See SALMON (G.).

MACROBERT (T. M.). See GRAY (A.).

MATHEWS (G. B.). See GRAY (A.).

PHILLIPS (H. B.). Differential equations. New York, Wiley, 1922. 78 pp.

DE POORTER (A.) et ALLIAUME (M.). Catalogue des manuscrits mathématiques et astronomiques de la Bibliothèque de Bruges. Bruges, 1922. 8vo. 50 pp.

Römert (J.). Der elementare Beweis des Fermatschen Satzes $x^{2n+1} + y^{2n+1} = z^{2n+1}$ auf Grund der Zerlegung in Faktoren und der Regeln der Potenzlehre. Halberstadt, J. Schimmelberg, 1920.

Rothe (H.). Vorlesungen über höhere Mathematik. Wien, Seidel, 1921. 11 + 691 pp.

Salmon (G.) und Fiedler (W.). Analytische Geometrie des Raumes. Unter Mitwirkung von A. Brill neu herausgegeben von K. Kommerell. 5te Auflage. Teil 1, Lieferung 1. Leipzig, Teubner, 1922. 10 + 366 pp.

Schütze (H.). Ueber das logarithmische Potential einer materiellen Fläche, deren Randkurve durch Inversion einer Parabel entstanden ist. (Diss., Halle.) Halle, 1919. 46 pp.

Singer (C.). Studies in the history and method of science. Edited by Dr. Charles Singer. Volume 2. Oxford, Clarendon Press, 1921. 22 + 559 pp. + 55 plates.

Soper (H. E.). The numerical evaluation of the incomplete B-function. (Tracts for Computers, No. 7.) London, Cambridge University Press, 1921. 53 pp.

van Thijn (A.). Inleiding tot de hoogere analyse. Groningen, J. B. Wolters, 1920. 256 pp.

de Vries (H.). Leerboek der differentiaal- en integraalrekening en van de theorie der differentiaalvergelijkingen. Tweede deel. Groningen, Noordhoff, 1920. 8vo. 445 pp.

Wieleitner (H.). Geschichte der Mathematik. Teil 2: Von Cartesius bis zur Wende des 18ten Jahrhunderts. 2te Hälfte: Geometrie und Trigonometrie. Berlin, Vereinigung wissenschaftlicher Verleger, 1921.

II. ELEMENTARY MATHEMATICS

Aubert (P.) et Papelier (G.). Exercices de calcul numérique. Paris, Vuibert, 1920–1921. Tome 1, 188 pp. Tome 2, 252 pp.

Bisacre (F. F. P.). Applied calculus. An elementary text book. London, Blackie, 1921. 16 + 446 pp.

Busch (E.). Der Dreher als Rechner. Wechselräder-, Touren-, Zeit- und Konusberechnung in einfachster und anschaulichster Darstellung. Berlin, Springer, 1919.

Düsing (K.). Die Elemente der Differential- und Integralrechnung in geometrischer Methode dargestellt. Leipzig, 1920.

Few (H. P.). Elementary determinants for electrical engineers. New York, Van Nostrand, 1922. 6 + 98 pp.

Gautier (—.). L'unité d'angle logique et le radian. Paris, Gauthier-Villars, 1921. 6 + 20 pp.

Houel (—.). Tables de logarithmes à cinq décimales. Paris, Gauthier-Villars, 1921.

Humbert (E.). Problèmes de trigonométrie. Paris, Vuibert, 1921. 8vo. 332 pp.

Jessop (C. M.). Elementary analysis. Cambridge, University Press, 1921. 8 + 175 pp.

Junker (E.). Differentialrechnung. 2te, verbesserte Auflage. Berlin, 1920.

Lorentz (H. A.). Lehrbuch der Differential- und Integralrechnung. 4te Auflage. Leipzig, 1922. 6 + 602 pp.

Murdoch (W. H. F.). Mathematical synopsis: algebra, logarithms trigonometry and spherical trigonometry. London, Bowman, 1922. 32mo. 42 pp.

d'Ocagne (M.). Vue d'ensemble sur les machines à calculer. Paris, Gauthier-Villars, 1922. 8vo. 69 pp.

Papelier (G.). See Aubert (P.).

Poirée (J.). Méthodes pour résoudre les problèmes de géométrie. Auch, Cocharaux, 1921. 48 pp.

Santz (A.). Santz-Multiplikator. Kleinste, das gesamte Zahlenreich umfassende Rechentafel. Berlin, Springer, 1920.

Shireby (R. M.). The slide rule applied to commercial calculations. London, Pitman, 1922. 72 pp.

Turner (J.). Practical mathematics; problems and questions. London and Edinburgh, McDougall, 1922. 80 + 10 pp.

Witting (A.). Einführung in die Infinitesimalrechnung. Band 2: Integralrechnung. 2te Auflage. Leipzig, 1921.

III. APPLIED MATHEMATICS

Abraham (M.). Theorie der Elektrizität. Band 1: Einführung in die Maxwellsche Theorie der Elektrizität. Mit einem einleitenden Abschnitt über das Rechnen mit Vektorgrössen in der Physik. Von A. Föppl. 6te, umgearbeitete Auflage, herausgegeben von M. Abraham. Leipzig, 1921.

Ariès (E.). L'œuvre scientifique de Sadi Carnot: Introduction à l'étude de la thermodynamique. Paris, Payot, 1921. 160 pp.

Baker (G. S.). Ship form, resistance, and screw propulsion. London, Constable, 1921.

Baye (J.). See Tilgher (A.).

Buchholz (H.). Theorie und Berechnung der statisch unbestimmten Tragwerke. Elementares Lehrbuch. Berlin, 1921.

Carvallo (J.). Traité d'électricité théorique en vue d'applications industrielles. Paris, Dunod, 1922. 12 + 548 pp.

Duffing (G.). Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung. Braunschweig, Vieweg, 1918. 6 + 134 pp.

Einstein (A.). Die Grundlagen der allgemeinen Relativitätstheorie. Leipzig, 1921.

——. Sidelights on relativity. I: Ether and relativity. II: Geometry and experience. Translated by G. B. Jeffery and W. Perrett. London, Methuen, 1922. 4 + 56 pp.

Esclangon (E.). Les preuves astronomiques de la relativité. Paris, Gauthier-Villars, 1922. 8vo. 28 pp.

Föppl (A.). See Abraham (M.).

GEIGER (M.). Die philosophische Bedeutung der Relativitätstheorie. Halle, 1921.

HINKS (A. R.). Map projections. 2d edition. Cambridge, University Press, 1921. 12 + 158 pp.

HOTOPP (L.). See KECK (W.).

INSOLERA (F.). Lezioni di matematica finanziaria. Torino, Fratelli Treves, 1921. 2 volumi.

JAEGER (W.). Elektrische Messtechnik. 2te Auflage. Leipzig, Barth, 1922. 22 + 528 pp.

JEFFERY (G. B.). See EINSTEIN (A.).

KECK (W.). Mechanik elastisch fester und flüssiger Körper. 4te, Auflage, bearbeitet von L. Hotopp. Hannover, 1919.

MISSIROLI (M.). See TILGHER (A.).

MÖLLER (M.). Kraftarten und Bewegungsformen. Die äusseren Bewegungen mit einführender Aufgaben-Sammlung. Braunschweig, Vieweg, 1922. 8 + 148 pp.

NEUMANN (E. R.). Vorlesungen zur Einführung in die Relativitätstheorie. Jena, Fischer, 1922.

OLLIVIER (H.). Cours de physique générale. 2e édition, entièrement refondue. Tomes 1–2. Paris, Hermann, 1921–1922. 760 + 416 pp.

DU PASQUIER (L. G.). Le principe de la relativité et les théories d'Einstein. Paris, Doin, 1922. 8vo. 16 + 511 pp.

PERRETT (W.). See EINSTEIN (A.).

PLANCK (M.). Physikalische Rundblicke. Gesammelte Reden und Aufsätze. Leipzig, 1922.

——. Das Wesen des Lichts. 2te, umgearbeitete Auflage. Berlin, 1920.

VON ROHR (M.). Die Brille als optisches Instrument. 3te Auflage. Berlin, 1921.

ROUGIER (L.). En marge de Curie, de Carnot, et d'Einstein. Nouvelle édition. Paris, Chiron, 1920. 16mo.

ROY (L.) Cours de mécanique rationnelle. Paris, Gauthier-Villars, 1921. 8vo. 6 + 260 pp.

STÄCKEL (P.). Bemerkungen zum Prinzip des kleinsten Zwanges. (Heidelberger Akademie der Wissenschaften.) Heidelberg, Carl Winter, 1919.

TILGHER (A.). Relativistes contemporains. Préface de M. Missiroli. Traduction de la 3e édition italienne par J. Baye. Paris, Librairie Farine, 1922. 8vo. 88 pp.

VAHLEN (T.). Ballistik. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 8vo. 12 + 231 pp.

VINCENT (M.). Essai sur les principes des théories physiques, suivi d'une théorie dynamique de la masse. Paris, Librairie Fischbacher, 1921. 16mo. 276 pp.

WARBURG (H. D.). Tides and tidal streams; a manual for the use of seamen. Cambridge, University Press, 1922. 7 + 95 pp.

WITTE (H.). Raum und Zeit im Lichte der neueren Physik. Eine allgemeinverständliche Entwicklung des raumzeitlichen Relativitätsgedankens bis zum Relativitätsprinzip der Trägheitssysteme. 3te Auflage. Braunschweig, Vieweg, 1920. 4 + 88 pp.

THE TWENTY-NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The twenty-ninth summer meeting of the Society, consisting of two sessions, was held at the University of Rochester, Rochester, New York, Thursday and Friday, September 7–8, 1922, in conjunction with the meeting of the Mathematical Association of America. Many members of the Society attended the special session of the Association on Thursday morning, held at the Research Laboratory of the Eastman Kodak Company, and members of both organizations were invited to visit the buildings of the Bausch and Lomb Optical Company on Friday afternoon. A much appreciated feature of the meeting was the chamber concert given in Kilbourn Hall of the new Eastman School of Music which is an integral part of the University of Rochester. The joint dinner, preceded by a very enjoyable automobile ride arranged by the alumni of the University, was held on Thursday evening at the Oak Hill Country Club, on the future site of the University. At this dinner Professor R. C. Archibald, President of the Association and Librarian of the Society, gave an interesting talk on his impressions of European conditions.

The attendance at the scientific sessions included the following seventy members of the Society:

Archibald, W. J. Berry, William Betz, Borger, W. G. Bullard, R. W. Burgess, Cairns, W. B. Carver, Clarke, H. E. Cobb, Coble, Comstock, Copeland, Crathorne, Decker, Durfee, Eisenhart, Ettlinger, Finkel, Fischer, W. B. Ford, Gale, Gilman, Gummer, Harrell, E. R. Hedrick, Hildebrandt, Hurwitz, Ingels, Karpinski, Kellogg, Kuhn, W. D. Lambert, Lefschetz, Lennes, Long, Matheson, G. A. Miller, Norman Miller, Mirick, C. N. Moore, F. H. Murray, F. W. Owens, H. B. Owens, Pitcher, Plant, Rasor, F. W. Reed, Reid, Harris Rice, R. G. D. Richardson, Ritt, E. D. Roe, J. R. Roe, Seely, Sharpe, Sherk, W. G. Simon, Slaught, C. E. Smith, Virgil Snyder, Swartzel, Watkeys, J. H. Weaver, Wedderburn, D. E. Whitford, F. B. Williams, W. L. G. Williams, Yeaton, J. W. Young.

The Secretary announced the election of the following persons to membership in the Society:

Dr. Beulah May Armstrong, University of Illinois;
Mr. Santiaga Artiaga, City Engineer, Manila;
Professor Francis Easton Carr, Oberlin College;
Dr. Vishnu D. Gokhale, University of Chicago;
Dr. Charles Jordan, Budapest;
Professor Henri Lebesgue, Collège de France;
Mr. Earle Brenneman Miller, University of Wisconsin;
Mr. Edward Everett Rhodes, Mutual Benefit Life Insurance Company;
Mr. Harvey Alexander Simmons, University of Michigan;
Dr. Ernest Bloomfield Zeisler, University of Chicago.

At the meeting of the Council, twenty-one applications for membership in the Society were received.

The Council announced the appointment of the following committees: Professors Haskins, Fiske and H. S. White on award of the Bôcher Memorial Prize; Professors Curtiss, Dunham Jackson and Mitchell on nomination of officers; Professors Coolidge, Fischer, Kellogg, Roever, Tyler, and the Secretary of the Society, on arrangements for the annual meeting; Professors Fite and Dresden and Mr. Joffe on investment of the Eliakim Hastings Moore Fund.

It was announced that the extra volume of the TRANSACTIONS is being printed by the Eschenbach Company, of Easton, Pa.

Greetings were sent to the newly-formed Mathematical Society of Belgium.

At the meeting of the Society, the reciprocity agreement with the London Mathematical Society was ratified, on recommendation of the Council. The medal presented to the Society by the l'Académie royale des Sciences et Beaux-Arts de Belgique, on the occasion of its one hundred fiftieth anniversary, was exhibited. A resolution was passed thanking the University of Rochester, and in particular the department of mathematics, for its generous hospitality.

At the sessions of the Society Professor A. B. Coble presided, relieved by Professors L. P. Eisenhart, G. A. Miller and Virgil Snyder. The session of Friday morning was especially marked by a paper read, at the request of the programme committee, by Professor C. A. Fischer, on *Functions of lines*.

Titles and abstracts of the papers read at this meeting follow below. The papers of Professors Cole, R. L. Moore and Kline, Dr. Schmidt, Professor Myller, Dr. Zeldin, Professor Graustein, Miss Carlson, Professor Evans, Dr. Franklin, Dr. Murray, Professor Dodd, and the third paper by Professor Schwatt, were read by title. Dr. Camp's paper was read by Professor C. N. Moore, Mr. Cowling's by Professor Ettlinger, Professor Schwatt's first paper by Professor Snyder, and his second by Professor Ettlinger. Mr. Cowling was introduced by Professor Ettlinger.

1. Professor L. P. Eisenhart: *Condition that a tensor be the curl of a vector.*

This paper will appear in the December number of this BULLETIN.

2. Professor S. Lefschetz: *A new class of topological invariants for two-sided manifolds.*

In a two-sided M_n consider a fundamental system $\Gamma_k{}^1$, $\Gamma_k{}^2$, . . . for the k-cycles, $k \leqq \frac{1}{2}n$, and a similar one Γ^1_{n-k}, Γ^2_{n-k}, . . . for the $(n - k)$-cycles; then denote by $(\Gamma_k{}^i\Gamma^j_{n-k})$ the number of intersections of the two cycles when a definite sign is attached to each point of intersection after the manner of Poincaré. The elementary divisors of the matrix

$$||(\Gamma_k{}^i\Gamma^j_{n-k})||$$

are the invariants in question. It is readily seen that for an M_2 they are all equal to unity. Remarkably enough this is also true for an algebraic surface (though not for the most general M_4), also for a d-dimensional algebraic variety and $k = 1, 2$. As the topological invariants already known have rather arbitrary values for algebraic surfaces, it would seem that their relations with those just defined are unlikely to be simple.

3. Professors F. R. Sharpe and Virgil Snyder: *The* (1, 2) *quaternary correspondences associated with certain space involutions.*

In this paper Professors Sharpe and Snyder use the complex of lines determined by the pairs of conjugate points of a given space involution to find the (1, 2) quaternary correspondence associated with the involution. The method can be applied to known involutions that are not readily discussed by the inverse process. The non-monoidal cubic involution is proved to be rational. The special case of the cubic inversion is of particular interest because of the complicated configuration of the fundamental elements.

4. Professor C. N. Moore: *On the summability of the double Fourier's series.*

The principal result of this paper is the theorem that the double Fourier's series corresponding to a function of two variables $f(x, y)$ that is integrable (Lebesgue) in a certain region of the (x, y)-plane is summable $(C1)$ to the value of the function throughout the region, except for a set of points of measure zero. This is an extension to double Fourier's series of a theorem with regard to ordinary Fourier's series due to Lebesgue. A further result of the paper is an extension to double Fourier's series of Fejér's theorem under wider conditions than have hitherto been found.

5. Professor N. J. Lennes: *The theory of sets and the foundation of arithmetic.*

The commutative and associative laws of addition and multiplication as well as the distributive law of multiplication are easily shown to hold under any set of axioms sufficient for the ordinary theory of sets. In the last analysis, integers may be regarded as counters for "things," and hence it is proposed to make the properties of integers under the operations of addition and multiplication depend logically upon the corresponding properties of sets.

This paper gives an outline of the process by which the usual axioms of arithmetic may be derived as theorems from the corresponding propositions in the theory of sets.

6. Professor F. N. Cole: *Kirkman parades.*

This paper will appear in the December number of this BULLETIN.

7. Professors R. L. Moore and J. R. Kline: *On the definition of a simple closed surface.*

The authors suggest the following definition for a simple closed surface: A simple closed surface is a closed, bounded and connected point set M such that (a) every simple closed curve J which belongs to M is the common boundary, with respect to M, of two mutually separated connected subsets of M whose sum is $M - J$. (b) If P is a point of M and ϵ is a positive number, there exists in M a simple closed curve J such that, of the two mutually separated connected sets into which M is divided by the omission of J, one contains P and is of diameter less than ϵ. If in a euclidean space of three dimensions M is a point set satisfying this definition, then the authors prove that there is a one-to-one continuous correspondence between the point set M and the surface of a sphere.

8. Dr. Karl Schmidt: *The theory of functions of one Boolean variable.*

In this paper, the expression $f(x) = ax + b\bar{x}$ of the Boolean algebra, or algebra of logic, is studied by new methods, which lead to new results and throw new light on old results. The principal references are to the works of E. Schröder and Eugen Müller. The paper will be published in the TRANSACTIONS OF THIS SOCIETY.

9. Professor A. Myller: *Representation of rectilinear motion by the geodesics of a surface.*

Professor Myller shows that the world-lines, in the Minkowski sense, of any rectilinear motion are in one-to-one correspondence with the geodesics of a surface of revolution, and conversely.

10. Dr. S. D. Zeldin: *Note on steady fluid motion.*

This paper appeared in full in the July number of this BULLETIN.

11. Professor W. C. Graustein: *Real representations of analytic complex curves.*

In a previous paper, presented to the Society in December, 1919, all the representations of a complex point by pairs of ordered real points which satisfy certain general conditions were determined. In the present paper application is made to analytic complex curves, in the plane and in three-dimensional space, of certain of these representations, namely, those invariant under direct transformations of similarity.

12. Miss Elizabeth Carlson: *Extension of Bernstein's theorem to Sturm-Liouville sums.*

The following theorem is proved in this paper: The maximum of the absolute value of the derivative of a Sturm-Liouville sum of order n does not exceed nhM, where M is the maximum of the absolute value of the sum itself, and h is a constant independent of n and of the coefficients in the sum. The corresponding theorem for finite trigonometric sums is due to S. Bernstein.

13. Professor G. C. Evans: *A Bohr-Langmuir contact transformation.*

The author considers the relation between the Bohr and Langmuir atoms, the latter having a quantized force, with regard to the question as to whether the "events" of one system may be transformed into the events of the other. It is shown that one cannot be the transform of the other by any contact transformation strictly so-called, which does not involve the time explicitly, but that the Langmuir atom can be obtained from the Bohr atom by a transformation which does not involve the time and is a contact transformation "im kleinen," although not "im grossen." The vibrating Langmuir atom is seen to be equivalent to the Bohr atom with elliptical electron orbit; the completely static atom is equivalent to the Bohr atom with circular orbit.

14. Professor G. C. Evans: *An elementary theory of competition.*

The author discusses postulates for monopoly, cooperation and two different sorts of competition, and analyzes them with reference to a simple economic system. The order of prices is obtained, and the effects of price fixing, of excess profits taxation and of certain typical changes in the cost function.

15. Professor G. A. Miller: *Groups in which the number of operators in a set of conjugates is equal to the order of the commutator subgroup.*

The author first proves that every abelian group can be used as a commutator subgroup of some group which involves a set of conjugate operators whose number is equal to the order of the commutator subgroup. He then determines the properties of the system of groups such that every group of the system contains a set of conjugate operators whose number is equal to the order of the commutator subgroup diminished by one. In particular, he proves that every group of such a system must also contain a set of conjugate operators whose number is equal to the order of the commutator subgroup, and that this commutator subgroup is always an abelian prime power group of type $(1, 1, 1, \cdots)$.

16. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier expansions.*

By assuming u to be a product of functions of the separate variables the author finds a solution of $\Sigma_{i=1}^{p}\partial u/\partial x_i + \lambda u = 0$ for the region $-\pi \leqq x_i \leqq \pi$ and the boundary conditions which make u identical for both ends of the several x_i-intervals. This depends on the solution of p ordinary differential equations with boundary conditions, each containing a different parameter. The expansion of a function f in p variables, which is made up of a finite number of pieces, each real, continuous, and possessing continuous partial derivatives, involves solutions of the adjoint systems and is shown by an extension of Professor Birkhoff's method to converge uniformly to f inside the subregion of any such piece. At a point of discontinuity the series, which is always expressible as a multiple Fourier series, converges to the mean value. By a suitable interpretation this is also true on the boundaries. This is believed to be the first adequate proof of convergence for a multiple Fourier series in more than two variables.

17. Professor E. H. Clarke: *On the minimum of the sum of a definite integral and a function of a point.*

Hadamard, in his *Leçons sur le Calcul des Variations,* has given, in § 159, pp. 176–177, a short notice of this problem. It may be stated briefly as follows: Given two points P_1 and P_2 in the xy plane and two functions $\varphi(x, y)$ and $f(x, y, y')$ of class C_{111}, to find a curve C, joining P_1 to P_2, and a point $\pi(x, y)$, lying on the curve C, such that the sum of the numerical value of the integral of $f(x, y, y')$ taken along C and the numerical value of the function $\varphi(x, y)$ for the point π shall be as small as possible. It is proved in this paper that in addition to the usual necessary conditions of the calculus of variations for the arcs $\rho_1\pi$ and $\pi\rho_2$ there are certain other necessary conditions which must hold at the point π and for the family of broken extremals with corners on a curve K through π. These conditions, when slightly strengthened, are shown to be sufficient to insure the existence of a minimum for the above sum.

18. Professor H. J. Ettlinger: *A simple proof of a fundamental lemma concerning the limit of a sum.*

The author gives a proof of the following lemma upon which may be based the theory of the Riemann integral.

Let the interval $I : a \leqq x \leqq b$ be divided into n equal sub-intervals, $I(i, n)$, each of length $\Delta_n x = (b - a/n)$. To each $I(i, n)$ there corresponds a number $h(i, n)$. Let $|h(i, n)| \leqq M$ for all values of $i \leqq n$ and n, where M is a constant. If x is any fixed value in I, then for each value of n, x is contained in at least one of the sub-intervals. Let it be designated by $I(x, n)$. For each fixed x and each subdivision $I(x, n)$ let the corresponding number be $h(x, n)$, and let $\lim_{n \to \infty} h(x, n) = 0$. Then $\lim_{n \to \infty} \sum_1^n h(i, n)\Delta_n x = 0$.

The proof makes use of simple limit theorems such as are to be found in every calculus textbook.

19. Mr. A. H. Cowling: *Application of Duhamel's theorem to the convergence proof for approximate solutions of differential equations.*

Runge, Heun and Kutta have given certain formulas for the approximate solution of the differential equation $y' = f(x, y)$. Assuming the conditions for which the Cauchy polygon converges, Mr. Cowling shows that these approximations approach the true solution as a limit. In this proof, use is made of Duhamel's theorem in the form given by Professor Ettlinger (see forthcoming paper in the AMERICAN MATHEMATICAL MONTHLY).

20. Dr. Philip Franklin: *Two theorems on multiple integrals.*

This paper will appear in the December number of this BULLETIN.

21. Professor E. L. Dodd: *An extension of the theorem of Bayes, by the use of a certain limit.*

If an observed event E must have been preceded by one of the mutually exclusive conditions C_i, $i = 1, 2, \cdots$; and $f(i)$ is the *a priori* probability that C_i would obtain; and $p(i)$ is the "productive" probability that C_i would yield E; then the *a posteriori* probability $P(r)$ that C_r preceded E is $f(r)p(r) \div \Sigma f(i)p(i)$. If, however, $f(x)dx$ is the probability that an error will lie between x and $x + dx$, and all real values are postulated as equally likely *a priori*, then $f(x) \equiv 0$,—if, indeed, it can be defined at all. The following extension is suggested to meet this difficulty:

$$P(r) = \lim_{t \to \infty} \frac{f(r,\ t)p(r)}{\sum_{i=-t}^{i=t} f(i,\ t)p(i)}.$$

Integration may replace summation. The author gives an example justifying this extension.

22. Dr. W. L. G. Williams: *Fundamental systems of protomorphic formal modular seminvariants of binary forms.*

The author defines fundamental systems of formal modular seminvariants, modulo p, of one binary form and of several binary forms and finds fundamental systems in both cases. These are remarkable in that each of them contains only one formal modular seminvariant not congruent, modulo p, to an algebraic seminvariant. Sufficient conditions for the existence of certain algebraic invariants are derived from the theory of formal modular concomitants and a connection with Hermite's law of reciprocity is noted.

23. Professor O. D. Kellogg: *An example in potential theory.*

The question of the extreme of generality of which the Dirichlet principle is susceptible is closely connected with the question of finite discontinuities of harmonic functions, where by discontinuity is meant a discontinuity either of the function or of a first derivative. It is well known that a harmonic function of x and y cannot have a finite non-removable discontinuity at an isolated point. Professor Kellogg gives a typical example of a harmonic function having finite non-removable discontinuities at every point of a nowhere dense perfect point set of measure zero. Green's function for the infinite plane, bounded by this set, is also established.

24. Mr. F. H. Murray: *On certain systems of differential equations containing a parameter.*

Systems of differential equations, of the form

$$dx_i/dt = X_i(x_1, \cdots, x_n, v, t), \qquad (i = 1, 2, \cdots, n)$$

where X_i is analytic in v in the neighborhood of v_0 and satisfies the usual conditions with respect to $x_1, \cdots, x_n, t$ in a certain domain $|t - t_0| < a$, $|x_i - x_i^0| < b$, have been discussed by Poincaré, Picard, and others. In this paper the writer proposes to give a similar discussion of the system

$$dx_i/dt = X_i(x_1, \cdots, x_n, w)$$

where X_i satisfies the usual conditions with respect to $x_1, \cdots, x_n$, and is periodic of period 2π with respect to the variable $w = vt$, for large values of v. An application is made to a problem related to the restricted problem of three bodies.

25. Mr. F. H. Murray: *Periodic solutions in the problem of three bodies.*

In *Les Méthodes Nouvelles de la Mécanique Céleste,* Poincaré gave a discussion of the existence of periodic solutions of the third kind in the problem of three bodies, which has formed the basis for later researches by Poincaré and other writers. It is the purpose of this paper to present an alternative demonstration of the existence of these solutions, using certain results of von Zeipel.

26. Professor C. A. Fischer: *Functions of lines.*

The first part of this paper contains an elementary treatment of such concepts as the continuous functional, or, as it is sometimes called, function of a line, that is, curve, the linear functional and the derivative and differential of a functional. Then completely continuous transformations and Stieltjes integral equations are discussed briefly.

27. Professor I. J. Schwatt: *The expansion of a certain function.*

The author shows how the expansion of

$$(a_1 \sin^{p_1} x + a_2 \cos^{p_2} x)^n,$$

in powers of x, where p_1 and p_2 are positive integers and n is any real number, depends on the expansion

$$\sin^{p_1} x \cos^{p_2} x = \frac{1}{2^{p_1+p_2}} \sum_{\kappa=0}^{\infty} (-1)^\kappa \frac{x^{2\kappa+p_1}}{(2k+p_1)!} \sum_{\alpha=0}^{p_2} \binom{p_2}{\alpha} \times \sum_{\beta=0}^{p_1} (-1)^\beta \binom{p_1}{\beta} (p_1 + p_2 - 2\alpha - 2\beta)^{2\kappa+p_1}.$$

28. Professor I. J. Schwatt: *The summation of a family of deranged series.*

The author finds the sum of the series obtained by retaining throughout

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{a + nh}$$

groups of p successive terms, beginning with the first term, and omitting q successive terms after each of these groups. In this case the known tests fail to disclose the conditions under which the resulting series is convergent or divergent. These conditions are determined from the sum of the series. The series is convergent for all values of p and q, except when p and q are both odd, in which case the series is divergent.

29. Professor I. J. Schwatt: *The sum of the harmonic series.*

The author has developed an expression for $Q_{n,k}$, the sum of the products of 1, 2, 3, $\cdots$, n taken k at a time, which he believes to be new.

$$Q_{n,k} = \prod_{\alpha=1}^{n} \Big(\sum_{k_\alpha = k_{\alpha-1}+1}^{n-k+\alpha} k_\alpha \Big), \qquad k_\alpha = 0,$$

$$= \frac{(-1)^k}{n!} \binom{n}{k} (n+1+k)!$$

$$\sum_{\alpha=1}^{k} \frac{1}{(n+1+\alpha)(k+\alpha)!} \binom{k}{\alpha} \sum_{\beta=1}^{\alpha} (-1)^\beta \binom{\alpha}{\beta} \beta^{k+\alpha};$$

then

$$Q_{n,n-1} = n! \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n-1} \frac{n+k}{k!} \binom{2n}{n-k-1} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \alpha^{n-1+k}.$$

By means of this result the sum of the harmonic series is expressed in powers of n.

R. G. D. RICHARDSON,
Secretary.

A PROPERTY OF CERTAIN FUNCTIONS WHOSE STURMIAN DEVELOPMENTS DO NOT TERMINATE*

BY O. D. KELLOGG

Let $[u_k(x)]$ be a set of continuous Sturmian functions defined on the interval (a, b), i.e., solutions of an equation of the form

$$\frac{d}{dx}[k(x)\frac{d}{dx}u(x)] + \lambda g(x)u(x) = 0, \tag{1}$$

each satisfying two linear homogeneous self-adjoint boundary conditions and corresponding to a value of λ for which this is possible. We assume that the coefficients of the differential equation have derivatives of all orders, and that $g(x)$ does not vanish in the closed interval (a, b), nor $k(x)$ on the open interval. Let $f(x)$ denote a function with derivatives of all orders, satisfying boundary conditions to be specified presently. We proceed to call attention to a property which such functions must have if their developments in series of the functions $u_k(x)$ are not to terminate.

We denote by a_k the kth generalized Fourier coefficient of f:

$$a_k = \int f u_k g dx, \tag{2}$$

where we have omitted argument x, and limits of integration, a and b. No ambiguity will result from the abbreviation. If in the integral, we replace $u_k g$ by its value obtained from the differential equation (1), and integrate by parts, we obtain

$$a_k = -\frac{1}{\lambda_k}\int (kf')' u_k dx, \tag{3}$$

where the integrated terms have been omitted on the assumption that f satisfies the same self-adjoint boundary conditions as the u_k. Under this assumption, they vanish. We define a series of functions as follows:

$$f_n = -\frac{(kf'_{n-1})'}{g}, \qquad f_0 = f. \tag{4}$$

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* Presented to the Society, Feb. 25, 1922.

We now subject f to the hypothesis that it, and these derived functions, all satisfy the same boundary conditions as the u_k. It then appears that for all n,

$$a_k = \frac{1}{\lambda_k{}^n} \int f_n u_k g dx. \tag{5}$$

This is a simple generalization of the long familiar equation for the coefficients of a Fourier series. It is rather in the inference drawn from it, than in the generalization itself, that the interest lies. From the equation (5) we pass to an inequality. Let B_k denote the maximum of $|(b - a)g(x)u_k(x)|$, and F_n the maximum of $|f_n(x)|$. Then, evidently,

$$|a_k| \leqq B_k F_n / |\lambda_k|^n, \qquad \text{or} \qquad F_n \geqq (|a_k|/B_k)|\lambda_k|^n. \tag{6}$$

It follows that *unless the development of* $f(x)$ *terminates,* F_n *must, for all* n *greater than a determinable number* N, *exceed any exponential function of* n, Ae^{pn}, because of the known property of the characteristic numbers λ_k of having infinity as the only limit point of their absolute values.

In other words, *if* F_n *is less than any such exponential function, for positive* A *and* p *and for infinitely many values of* n, $f(x)$ *is a homogeneous linear function of the* $u_k(x)$ *with constant coefficients.*

This property takes on special interest when k and g are constant, for in this case the f_n are proportional to derivatives of f. With suitable boundary conditions, it then takes the form: *if the periodic function* $f(x)$ *has derivatives of all orders, it is either a trigonometric polynomial, or else the maximum of the absolute value of its nth derivative exceeds any exponential function* Ae^{pn} *for all* n *from a certain one on.*[*]

In the case of analytic functions of a complex variable, we have the result:

Either $f(z)$ *is a polynomial, or else the maximum of the absolute value of its nth derivative on any circle lying entirely within its domain of analyticity exceeds any exponential function* Ae^{pn} *from a given* n *on.*

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[*] The result in this particular form was announced to the Society Dec. 2, 1911. See this BULLETIN, vol. 18, p. 234.

RELATING TO THE PROOF OF AN EXISTENCE THEOREM FOR A CERTAIN TYPE OF BOUNDARY VALUE PROBLEM*

BY H. T. DAVIS

1. *Introduction.* The problem of proving the existence of a solution of an ordinary, linear, homogeneous differential equation of nth order which satisfies n linear, homogeneous boundary conditions of $(n - 1)$st order at p points has been attacked in various ways since it was first formulated by Sturm in 1836.† One of the most fruitful of these methods was that first employed by Bôcher,‡ and later applied by Birkhoff § and Ettlinger¶ to general self-adjoint systems of second order. This method is essentially the study of the zeros of a certain determinant of nth order by means of the properties of a system for which the existence of solutions has already been established.

The object of the present paper is to consider the problem by means of integral equations, from which is obtained a criterion of some elegance, although the generality of this method is somewhat limited by the way in which the parameter enters the integral equation.

2. *Theorems to be used.* Suppose $L(u)$ is any linear, homogeneous, ordinary differential expression of nth order and suppose the coefficient of $u^{(n)}(p_0)$ is different from zero in the interval $a \leqq x \leqq b$. Consider, then, the two systems:

$$(\mathrm{I}) \quad \begin{cases} L(u) + \lambda u = 0, \\ \quad U_i(u) = 0, \end{cases} \qquad (\mathrm{II}) \quad \begin{cases} L(u) + \lambda u = 0, \\ \quad V_i(u) = 0, \end{cases}$$

$(i = 1, 2, 3, \cdots, n)$, where U_i and V_i are linear combinations of u and its first $(n - 1)$ derivatives at the two points a and b.

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* Presented to the Society, April 15, 1922.

† Journal de Mathématiques, vol. 1 (1836), pp. 106–186.

‡ Comptes Rendus, vol. 140 (1905), p. 928.

§ Transactions of this Society, vol. 10 (1909), pp. 259–270.

¶ Transactions of this Society, vol. 19 (1918), pp. 79–96.

We shall suppose that the existence of an infinite sequence of λ's $(\lambda_1, \lambda_2, \cdots)$ is known for system I. Our object is to show how this fact may be used to find conditions upon the coefficients of system II which will insure the existence of a sequence of λ's $(l_1, l_2, \cdots)$ for this system also.

We make use of the following theorem in the theory of integral equations:

If the Green's function of the differential expression $L(u)$ *for such a set of boundary conditions as* I *or* II *is taken as the kernel of an integral equation of the second kind,*

$$f(x) = \phi(x) - \lambda \int_a^b K(x, t)\phi(t)dt,$$

the Green's function of the differential expression $L(u) + \lambda u$ *which corresponds to the boundary conditions* I *or* II *is the resolvant function* $K(x, t)$ *of this integral equation.*[*]

We recall that if

$$(1) \qquad g(x, t) = \pm \frac{\begin{vmatrix} u_1(t) & \cdots & u_1^{(n-2)}(t) & u_1(x) \\ u_2(t) & \cdots & u_2^{(n-2)}(t) & u_2(x) \\ \cdot & \cdot & \cdot & \cdot \\ u_n(t) & \cdots & u_n^{(n-2)}(t) & u_n(x) \end{vmatrix},}{2p_0(t)W(t)}$$

where $W(t)$ is the Wronskian of the fundamental set of solutions, $u_1, \cdots, u_n$, of the equation $L(u) = 0$, and their derivatives to the $(n - 1)$st order, and where the plus sign refers to $x \leqq t$, and the minus sign to $x \geqq t$, then

$$G(x, t) = g(x, t) + c_1u_1(x) + \cdots + c_nu_n(x)$$

is the Green's function of $L(u)$ provided $c_1, \cdots, c_n$ are so chosen that $G(x, t)$ will satisfy the boundary conditions $U_i(u) = 0$.

We notice that $G(x, t)$ exists and is a continuous function in both arguments except when

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[*] Hilbert, *Grundzüge einer allgemeiner Theorie der linearen Integralgleichungen*, Teubner, 1912, p. 49. Here the theorem is proved only for the special self-adjoint case of second order, but a consideration shows that the same proof may be extended to include the case of a system of nth order.

$$(2) \qquad F(\lambda) = \begin{vmatrix} U_1(u_1) & \cdots & U_1(u_n) \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ U_n(u_1) & \cdots & U_n(u_n) \end{vmatrix} = 0.$$

3. *Interrelation of Problems* I *and* II. We shall now suppose that $\Gamma_1(x, t)$ and $\Gamma_2(x, t)$ are the Green's functions of $L(u) + \lambda u$ for boundary conditions I and II, and that $K_1(x, t)$ and $K_2(x, t)$ are the Green's functions of $L(u)$ for the same boundary conditions. Consequently Γ_1 and Γ_2 are the resolvant functions for the integral equations

$$u(x) = \lambda \int_a^b K_1(x, t)u(t)dt, \qquad u(x) = \lambda \int_a^b K_2(x, t)u(t)dt,$$

and we are led to consider the two Fredholm determinants D_1 and D_2 which belong to problems I and II.

By the well known property of these determinants,* we have:

$$\frac{dD_1}{d\lambda} = - D_1 \int_a^b \Gamma_1(x, x)dx, \qquad \frac{dD_2}{d\lambda} = - D_2 \int_a^b \Gamma_2(x, x)dx.$$

Since the zeros of $D_1(\lambda)$ and $D_2(\lambda)$ are the characteristic values $(\lambda_1, \lambda_2, \cdots)$ and $(l_1, l_2, \cdots)$ for which solutions of I and II exist, we consider the ratio D_1/D_2. Between any two characteristic values λ_i and λ_{i+1}, D_1/D_2 is a continuous function providing D_2 does not vanish in the interval. Hence its derivative must vanish in the interval:

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{D_1}{D_2}\right) &= \frac{D_2 \dfrac{dD_1}{d\lambda} - D_1 \dfrac{dD_2}{d\lambda}}{D_2{}^2} \\ &= \frac{- D_2 D_1 \displaystyle\int_a^b \Gamma_1 dx + D_1 D_2 \int_a^b \Gamma_2 dx}{D_2{}^2} \\ &= - \frac{D_1}{D_2} \int_a^b [\Gamma_1(x, x) - \Gamma_2(x, x)]dx. \end{aligned}$$

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* Hilbert, loc. cit., pp. 12–13.

But suppose that it is shown that $\int_a^b(\Gamma_1 - \Gamma_2)dx \neq 0$ for $\lambda_i \leqq \lambda \leqq \lambda_{i+1}$. Our hypothesis that D_1/D_2 is a continuous function is contradicted. Consequently D_2 must vanish in the interval. This result may be stated in the following theorem.

THEOREM. *If the existence of a sequence of λ's for which solutions of problem* I *exist has been established, then a sequence of l's for problem* II *alternating or coincident with the λ's of problem* I *exist provided*

$$\int_a^b [\Gamma_1(x, x) - \Gamma_2(x, x)]dx \neq 0, \quad (\lambda_0 \leqq \lambda \leqq \lambda_m),$$

where Γ_1 *and* Γ_2 *are the Green's functions belonging to problems* I *and* II.

4. *Examples.* Two examples serve to illustrate the theorem:

$$u'' + \lambda u = 0,$$

$$(1) \qquad \text{I.} \quad \begin{cases} u(0) = 0, \\ u(1) = 0, \end{cases} \qquad \text{II.} \quad \begin{cases} u(0) = 0, \\ u'(1) = 0. \end{cases}$$

Specialize u_1 and u_2 by assuming $u_1(0) = 0$, $u_1'(0) = 1$; $u_2(0) = 1$, $u_2'(0) = 0$. Then

$$\int_0^1 (\Gamma_1 - \Gamma_2)dx = -\int_0^1 \frac{u_1^2(x)}{u_1(1)u_1'(1)}dx$$

which is never zero. Consequently the characteristic numbers of the two systems must always alternate with one another, a conclusion easily verified by explicitly solving the two systems.

$$(pu')' + qu + \lambda u = 0,$$

$$(2) \qquad \text{I.} \quad \begin{cases} u(a) = 0, \\ u(b) = 0, \end{cases} \qquad \text{II.} \quad \begin{cases} u(a) - u(b) = 0. \\ u'(a) - u'(b) = 0. \end{cases}$$

Again specialize the fundamental system by assuming $u_1(a) = 0$, $u_1'(a) = 1$, $u_2(a) = 1$, $u_2'(a) = 0$.

Then

$$\int_a^b (\Gamma_1 - \Gamma_2)dx = \frac{1}{D_1D_2}\int_a^b \{[u_2^2(b) - (1 + p_1)u_2(b) + p_1]u_1^2(x) - [2u_1(b)u_2(b) - (p_1 + 1)u_1(b)]u_1(x)u_2(x) + u_1^2(b)u_2^2(x)\}dx,$$

where D_1 and D_2 are the characteristic functions (2) of the two systems and $p_1 = p(a)/p(b)$. A sufficient condition, then, that the characteristic numbers of the two systems shall either alternate or coincide, is that the quadratic form in $u_1(x)$, $u_2(x)$ shall be definite. But the discriminant Δ of the form is

$$\Delta = (p_1 - 1)^2 u_1^2.$$

Consequently the form will be definite if and only if we have $p(a) = p(b)$, which is the well known condition that system II shall be self-adjoint.*

THE UNIVERSITY OF WISCONSIN

NOTE CONCERNING THE ROOTS OF AN EQUATION

BY K. P. WILLIAMS

Professors Carmichael and Mason have published the following theorem.†

All roots of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{1}$$

are, in absolute value, less than

$$\sqrt{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}. \tag{2}$$

It is apparent that this limit may be greatly in excess of the actual maximum of the absolute values of the roots. An illustration of this fact is furnished by the equation

$$x^n + x^{n-1} + \cdots + x + 1 = 0. \tag{3}$$

The theorem asserts that $\sqrt{n+1}$ is greater than the absolute value of any root. If n is large this is rather meager and inexact information, since all roots are in absolute value exactly 1, irrespective of the value of n.

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* *Note on the roots of algebraic equations*, this BULLETIN, vol. 21 (1914), p. 21.

† This example is treated by Bôcher by different means in his *Leçons sur les Méthodes de Sturm*, 1917, pp. 83-91.

It is possible to modify in a very simple way the work of the authors quoted, and to obtain a theorem that in some cases gives much more exact knowledge. The work will merely be sketched in order to avoid unnecessary repetition of the work of the paper in question. It is shown in that paper that all roots of the given equation are not greater in absolute value than

$$\limsup_{m\to\infty} \sqrt[m]{C_m},$$

where

$$C_m = -\begin{vmatrix} 1 & 0 & 0 & \cdots & a_1 \\ a_1 & 1 & 0 & \cdots & a_2 \\ a_2 & a_1 & 1 & \cdots & a_3 \\ a_3 & a_2 & a_1 & \cdots & a_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The theorem cited is then derived by applying to this determinant the theorem of Hadamard relative to a maximum value for a given determinant.

Before applying the theorem of Hadamard it is evidently possible to modify the determinant in various ways, and a new theorem will result from each modification. Let us subtract the first column from the last, the second from the first, the third from the second, etc. We have then the determinant

$$C_m = -\begin{vmatrix} 1 & 0 & \cdots & a_1 - 1 \\ a_1 - 1 & 1 & \cdots & a_2 - a_1 \\ a_2 - a_1 & a_1 - 1 & \cdots & a_3 - a_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n - a_{n-1} & \cdot & \cdots & -a_n \\ -a_n & a_n - a_{n-1} & \cdots & \\ 0 & -a_n & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \end{vmatrix}$$

assuming $m > n$, the determinant being of order m. The application of the theorem of Hadamard then gives the theorem:

THEOREM. *All roots of the equation* (1) *are less in absolute* *value than the quantity*

$$(4) \qquad \sqrt{1+|a_1-1|^2+|a_2-a_1|^2+\cdots+|a_n-a_{n-1}|^2+a_n{}^2}.$$

If we apply this theorem to equation (3) we see that all roots are less in absolute value than $\sqrt{2}$, irrespective of the value of n.

The quantity in (4) will evidently be smaller than the quantity (2) in many cases. As another illustration, consider the equation

$$(5) \qquad x^n+\frac{x^{n-1}}{\sqrt{1}}+\frac{x^{n-2}}{\sqrt{2}}+\frac{x^{n-3}}{\sqrt{3}}+\cdots+\frac{x}{\sqrt{n-1}}+\frac{1}{\sqrt{n}}=0.$$

The application of (2) gives

$$\sqrt{1+1+\frac{1}{2}+\cdots+\frac{1}{n}},$$

as a superior limit for the absolute values of the roots. Now Euler's constant tells us that this is of practically the same order as $\sqrt{\log n}$. We therefore could draw no conclusion as to whether the roots of (5) remain within a certain circle which does not change with n.

If, on the other hand, we apply (4), we have

$$\sqrt{1+\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)^2+\cdots+\left(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right)^2+\frac{1}{n}},$$

as the superior limit. The quantity under the radical is less than

$$1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n}$$
$$=1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots$$
$$+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\frac{1}{n}=2.$$

We therefore see that no root of (5) will, in absolute value, exceed $\sqrt{2}$, no matter how great n may be.

Indiana University

THE COMPLETE EXISTENTIAL THEORY OF HURWITZ'S POSTULATES FOR ABELIAN GROUPS AND FIELDS*

BY B. A. BERNSTEIN

1. *Introduction.* Hurwitz has proposed sets of postulates for abelian groups and fields.† If F', F'', F_n denote his sets for denumerable, continuous, and finite fields respectively, G', G'', G_n the corresponding sets for abelian groups, then I have proved in another paper‡ the following theorem.

THEOREM A. *Postulate-sets* F', F'', G', G'', G_n $(n > 1)$ *are each completely independent; postulate-set* F_n *is completely independent§ when, and only when,* n *exceeds* 2 *and is a power of a prime.*

The object of this note is to investigate postulate-set F_n further, and to prove the following theorem, which, together with Theorem A establishes the *complete existential theory*§ of each of Hurwitz's six postulate-sets for abelian groups and fields.

THEOREM B. *For postulate-set* F_n, *when* n *exceeds* 2 *and is not a power of a prime, there exists no system having the character* $(++++++)$, *but there exist systems having all the other characters; when* $n = 2$ *there exist no systems having the characters* $(-+++-+)$ *and* $(-+-+-+)$, *but there exist systems having all the other characters.*

2. *Hurwitz's Postulates* F_n *for Finite Fields.* For finite fields Hurwitz's postulates are as follows (K, $\oplus$, $\odot$ being undefined):

(A_1) If a, b, c, $a \oplus b$, $c \oplus b$, and $a \oplus (c \oplus b)$ belong to K, then $(a \oplus b) \oplus c = a \oplus (c \oplus b)$.

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[*] Presented to the Society April 8, 1922.

[†] W. A. Hurwitz, *Postulate-sets for abelian groups and fields*, ANNALS OF MATHEMATICS, (2), vol. 15 (1913), p. 93.

[‡] *On complete independence of Hurwitz's postulates for abelian groups and fields*, ANNALS OF MATHEMATICS, (2), vol. 24 (1922).

[§] See E. H. Moore, *Introduction to a form of general analysis*, New Haven Mathematical Colloquium, Yale University Press, p. 82.

(A_2) If a and b belong to K, then there is an element x of K such that $a \oplus x = b$.

(M_1) If a, b, c, $a \odot b$, $c \odot b$, and $a \odot (c \odot b)$ belong to K, then $(a \odot b) \odot c = a \odot (c \odot b)$.

(M_2) If a and b belong to K, and $a \oplus a \neq a$, there is an element x of K such that $a \odot x = b$.

(D) If a, b, c, $a \odot b$, $a \odot c$, $b \oplus c$, $(a \odot b) \oplus (a \odot c)$ belong to K, then $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

(N_n) K contains $n > 1$ elements.*

3. *Proof of Theorem B.* The proof of Theorem B is obtained with the help of the table below.†

(1) Systems 1–32 of the table have the characters $(\pm\pm\pm\pm\pm -)$ for F_n.

(2) Let 1′–32′ be the systems obtained from 1–32 by substituting (a) the class of *least positive residues modulo n* for the class of integers and (b) the *least positive residue modulo n of* $a + b$, ab, $a - b$ for $a + b$, ab, and $a - b$ respectively, except that $a^2 + (b - 1)^2$ in 16 is left unchanged. Then when $n > 2$, systems 2′–32′ will have the characters $(\pm\pm\pm\pm\pm +)$ except $(++++++)$; when $n = 2$, systems 1′–9′, 11′–20′, 22′–32′ will be systems having all the characters $(\pm\pm\pm\pm\pm +)$ except $(-+++-+)$ and $(-+-+-+)$.

(3) When $n > 2$ and not a power of a prime there exists no field.

(4) When $n = 2$ there exists no system having the character $(-+++-+)$. For, since postulates (A_1) and (D) have to be contradicted, and (A_2) satisfied, $a \oplus b$ must be defined by the table

$\oplus$	0	1
0	1	0
1	1	0

Further, since postulates (M_1) and (M_2) have to be satisfied,

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[*] Postulate-sets F' and F'' are obtained from set F_n by substituting for (N_n) respectively: (N') K is countably infinite, (N'') K has the cardinal number of the continuum. Sets G_n, G', G'' are obtained from F_n, F', F'' respectively by omitting (M_1), (M_2), and (D).

[†] This table may also be used conveniently in the proof of Theorem A.

$a \odot b$ must be defined by one or the other of the tables:

$\odot$	0	1
0	1	0
1	0	1

$\odot$	0	1
0	0	1
1	1	0

That is, we must have either

$$a \oplus b = b + 1 \pmod 2, \qquad a \odot b = a + b + 1 \pmod 2$$

or else

$$a \oplus b = b + 1 \pmod 2, \qquad a \odot b = a + b \pmod 2.$$

But for either case postulate (D) would be satisfied.

(5) That when $n = 2$ there is no system having the character $(-+-+-+)$ I have shown in the paper cited above.

This completes the proof of our theorem.

SYSTEMS HAVING THE CHARACTERS $(\pm\pm\pm\pm\pm-)$ FOR F_n

No.	Character	K	$a \oplus b$	$a \odot b$
1	$(+++++-)$	Integers*	$a+b$	ab
2	$(++++--)$	“	$a+b$	$a+b$
3	$(+++-+-)$	“	$a+b$	0
4	$(++-++-)$	“	$a+b$	b
5	$(+-+++-)$	“	a	$a+b$
6	$(-++++-)$	“	b	$a+b$
7	$(+++---)$	“	$a+b$	1
8	$(++-+--)$	“	$a+b$	$b+1$
9	$(+-++--)$	“	0	$a+b$
10	$(-+++--)$	“	$a-b$	$a+b$
11	$(++--+-)$	“	$a+b+1$	$b+0/a$
12	$(+-+-+-)$	“	0	0
13	$(-++-+-)$	“	$b+1$	$a/0$
14	$(+--++-)$	“	0	b
15	$(-+-++-)$	“	b	b
16	$(--+++-)$	“	$b+\dfrac{0}{a^2+(b-1)^2}$	a
17	$(++----)$	“	$a+b$	$a+1$
18	$(+-+---)$	“	0	1
19	$(-++---)$	“	$b+1$	1
20	$(+--+--)$	“	0	$b+1$
21	$(-+-+--)$	“	$a-b$	$a-b$
22	$(--++--)$	“	$ab+a$	$a+b$
23	$(+---+-)$	“	1	$b+0/a$
24	$(-+--+-)$	“	$b+1$	$b+0/a$
25	$(--+-+-)$	“	$ab+a$	0
26	$(---++-)$	“	$a+1$	b
27	$(+-----)$	“	0	$a+1$
28	$(-+----)$	“	$b+1$	$a+1$
29	$(--+---)$	“	$a+1$	a
30	$(----+-)$	“	$b+0/a$	$b+0/a$
31	$(---+--)$	“	$ab+a$	$b+1$
32	$(------)$	“	$a+1$	$a+1$

* Positive, negative, and 0.

THE UNIVERSITY OF CALIFORNIA

ON KUMMER'S MEMOIR OF 1857 CONCERNING FERMAT'S LAST THEOREM*

BY H. S. VANDIVER

1. *Introduction.* In a previous paper under the same title,† the writer considered an article by Kummer,‡ and pointed out that the argument there used for proving certain results regarding the equation

$$x^\lambda + y^\lambda + z^\lambda = 0, \tag{1}$$

where x, y and z are integers and λ is an odd prime, is deficient and incorrect in several respects. Kummer attempts to prove four theorems which in my first paper were numbered I to IV. I pointed out that the proofs of Theorems I and IV are incomplete, and that the proofs of Theorems II and III are inaccurate. In the present paper additions to and modifications of Kummer's arguments will be given, by means of which the demonstrations Theorems I and IV will be completed.

2. *Proof of Theorem* I. Assume that h_1 is the first factor of the class number of the field $\Omega(\alpha)$, α being a primitive λth root of unity,

$$E_\nu(\alpha) = \prod_{k=0}^{\mu-1} e(\alpha^{\gamma^k})^{\gamma^{-2k\nu}},$$

$$e(\alpha) = \sqrt{\frac{(1-\alpha^\gamma)(1-\alpha^{-\gamma})}{(1-\alpha)(1-\alpha^{-1})}},$$

where γ is a primitive root of λ, and $\mu = (\lambda - 1)/2$; then Kummer's Theorem I may be stated as follows.

If h_1 is divisible by λ but not by λ^2, then one and only one Bernoulli number B_ν in the set B_i, $i = 1, 2, \cdots, \mu - 1$, is divisible by λ. If under this assumption we also have $h_2 \equiv 0 \pmod{\lambda}$, h_2 being the second factor of the class number of $\Omega(\alpha)$, then $E_\nu(\alpha)$ is the λth power of a unit in $\Omega(\alpha)$.

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* Presented to the Society October 30, 1920.

† PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 6, No. 5, pp. 266–69 (May, 1920).

‡ MATHEMATISCHE ABHANDLUNGEN OF THE BERLIN ACADEMY, 1857, pp. 41–74.

To complete Kummer's proof it is necessary to prove only the first statement in the theorem. Assume $\lambda > 5$. In another paper* the writer has given the relation

$$h_1 \equiv \prod_s \frac{\lambda(-1)^{(s\lambda^2-1)/2} B_{(s\lambda^2+1)/2}}{2^{(\lambda-3)/2}} \pmod{\lambda^2}, \tag{2}$$

$$(s = 1, 3, \cdots, \lambda - 2).$$

We shall now show that the assumption that two or more of the B's in the set B_i, $i = 1, 2, \cdots, \mu - 1$, are divisible by λ leads to the relation $h_1 \equiv 0 \pmod{\lambda^2}$ which will yield the proof desired, as it is known† that if $h_1 \equiv 0 \pmod{\lambda}$ then at least one of the B's is divisible by λ.

Assume that $B_a \equiv B_b \equiv 0 \pmod{\lambda}$, where a and b are each included in the set $1, 2, \cdots, \mu - 1$. Kummer‡ has shown that

$$\frac{B_c}{c} \equiv (-1)^{k\mu} \frac{B_{c+k\mu}}{c + k\mu} \pmod{\lambda},$$

where k is an integer and c is not a multiple of μ, and where $\lambda > 3$. This gives

$$(-1)^{s\mu(\lambda+1)} \frac{B_{(s\lambda^2+1)/2}}{(s\lambda^2 + 1)/2} \equiv \frac{B_{(s+1)/2}}{(s + 1)/2} \pmod{\lambda},$$

and if $(s + 1)/2 = a$, then s is included in the set $1, 3, 5, \cdots, \lambda - 2$, and the above congruence gives

$$B_{(s\lambda^2+1)/2} \equiv 0 \pmod{\lambda}.$$

Similarly

$$B_{(s_1\lambda^2+1)/2} \equiv 0 \pmod{\lambda},$$

where $(s_1 + 1)/2 = b$. Applying these relations to (2), we obtain $h_1 \equiv 0 \pmod{\lambda^2}$, and Theorem I is proved, as the remainder of the proof as set forth by Kummer, is, in my opinion, rigorous.

3. *Proof of Theorem* IV. We proceed to Kummer's fourth theorem:

THEOREM IV. *If h_1 is divisible by λ, but not by λ^2, and P is an ideal in $\Omega(\alpha)$, such that P^λ is the principal ideal $(F(\alpha))$,*

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* This BULLETIN, vol. 25, p. 460, relation (8) for $a = 2$.

† Vandiver, ibid., p. 461.

‡ CRELLE, vol. 41, p. 368.

$B_\nu \equiv 0 \pmod{\lambda}$, $\nu < \mu$, *then* P *is or is not a principal ideal according as*

$$\frac{d_0^{\lambda-2\nu} \log F(e^v)}{dv^{\lambda-2\nu}} \equiv 0 \text{ or } \not\equiv 0 \pmod{\lambda}.$$

In this statement, the notation $d_0^{\lambda-2\nu}$ means that the function $F(e^v)$ is to be differentiated $\lambda - 2\nu$ times with respect to v, and zero substituted for v in the result. The letter e denotes the Napierian base.

In my first note two criticisms of Kummer's proof of this theorem were made. I shall here modify his argument so as to dispose of the difficulties in question, and consequently complete the demonstration. The first of my criticisms referred to formula (A) on page 53 of Kummer's memoir. The number $\psi_r(\alpha)$ which appears there is defined in the eleventh line from the bottom of the page as the product of certain ideal factors, but this decomposition is proved to hold only for the case where $\psi_r(\alpha)$ contains ideals of degree not higher than the first. In another paper,* Kummer gives the corresponding formula for the generalized function $\psi_r(\alpha)$ which contains ideals of higher degree, as follows:

$$\text{(3)} \quad \text{ind } E_n(\alpha) \equiv \frac{\gamma^{2n} - 1}{2(1 + r^{\lambda-2n} - (r+1)^{\lambda-2n})} \cdot \frac{d_0^{\lambda-2n} \log \psi_r(e^v)}{dv^{\lambda-2n}} \pmod{\lambda},$$

where r is an integer, $1 < r < \lambda - 1$, $E_n(\alpha)$ and γ are defined as before, and ind $(E_n(\alpha))$ is i in the relation

$$(E_n(\alpha))^{(q^t-1)/\lambda} \equiv \alpha^i \pmod{\mathfrak{P}},$$

$\mathfrak{P}$ being an ideal prime factor of the odd prime q, and t the exponent to which q belongs modulo λ. Further

$$\psi_r(\alpha) = \alpha^{-(r+1)h + \text{ind}(g^h+1)},$$

where g is a primitive root of $\mathfrak{P}$ such that $g^{(q^t-1)/\lambda} \equiv \alpha \pmod{\mathfrak{P}}$, h ranges over the integers $0, 1, 2, \cdots, q^t - 2$, excepting $(q^t - 1)/2$, ind $(g^h + 1)$ being defined as i in the relation

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* CRELLE, vol. 44, p. 125.

$(g^h + 1) \equiv g^i \pmod{\mathfrak{P}}$. Although Kummer in defining q did not state that it was odd, his work is subject to that restriction, since his function $\psi_r(\alpha)$, as defined by him, has no meaning for $q = 2$, since in that case $(q^t - 1)/2$ is not integral. If, however, we take the function as defined by H. H. Mitchell * for the case $q = 2$, namely

$$\psi_r(\alpha) = \alpha^{-(r+1)h+\operatorname{ind}(g^h+1)}$$

where h ranges over the integers $1, 2, \cdots, 2^t - 2$, the formula (3) will also hold for this case, as Kummer's argument can be used without change except that it is necessary to note in proving formulas such as the following (loc. cit., middle of page 125)

$$(g^i - 1)(g^i - g^\lambda)(g^i - g^{2\lambda}) \cdots (g^i - g^{(\nu-1)\lambda}) \equiv 1 - g^{\nu i} \pmod{\mathfrak{P}},$$

that since $1 \equiv -1 \pmod{q}$ for $q = 2$,

$$g^{\nu i} - 1 \equiv 1 - g^{\nu i} \pmod{p}.$$

Kummer (loc. cit., page 120) gives the decomposition of the general $\psi_r(\alpha)$ into prime ideal factors. Instead of using this decomposition we shall examine the form of it given by Mitchell † and express the factorization of $\psi_r(\alpha)$, t arbitrary, in a form analogous to that given by Kummer for the case $t = 1$. Mitchell ‡ considers the Galois field of order q^t, where q is any prime, and $q^t - 1 = \lambda\nu$. Let the elements of this field other than zero each be represented as a power of a primitive root, and σ_i denote any element whose index is congruent to i, modulo λ. The symbol $\overset{j}{m_i}$ stands for the number of solutions of the relation $1 + \sigma_i = \sigma_j$ in the field. He then defines the function (page 167, relation 4),

$$\psi_{-a,-b}(\alpha) = \sum_{i,\,j} \overset{j}{m_i} \alpha^{-bi+(a+b)j},$$

where i and j each range over the integers $0, 1, \cdots, \lambda - 1$, and $a \not\equiv 0$, $b \not\equiv 0$, $a + b \not\equiv 0 \pmod{\lambda}$. For $b = -1$, $a = -r$,

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* TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 165.

† Loc. cit., p. 168.

‡ Loc. cit., p. 166, § 2.

this becomes

$$\psi_{r,1}(\alpha) = \sum_{i,\,j} \overset{j}{m}_i \alpha^{-(r+1)j+i}.$$

Since there are $\overset{i}{m}_j = \overset{j}{m}_i$ numbers $h \equiv j$ and ind $(g^h + 1) \equiv i$ (mod λ), the preceding relation may be written in the form

$$\psi_r(\alpha) = \Sigma\alpha^{-(r+1)h+\mathrm{ind}(g^h+1)}.$$

Now apply Mitchell's first theorem (loc. cit., page 173) to the particular function $\psi_r(\alpha)$. We have in this case $t = t_1$, where q belongs to the exponent t_1, modulo λ. We conclude that the number of times the ideal $\mathfrak{P}_i$, where this symbol designates the ideal obtained from $\mathfrak{P}$ by the substitution (α^k/α), $ki \equiv 1$ (mod λ), is equal to the number of the expressions $|-riq^{t-j}| + |-iq^{t-j}|$, $j = 0, 1, \cdots, t-1$, whose values exceed λ, the integer i assuming $(\lambda - 1)/t$ values prime to λ such that the quotient of no two of them is congruent modulo λ to a power of q, and the symbol $|x|$ denoting the least positive residue of x, modulo λ. Now if we select the integers l in the set $1, 2, \cdots, \lambda - 1$, which satisfy the relation $|-rl| + |-l| > \lambda$, we may show from what precedes that

$$\prod_c \mathfrak{P}_c = \psi_r(\alpha), \tag{4}$$

where c ranges over the integers which satisfy $cl \equiv 1$ (mod λ). For, the integers $1, 2, \cdots, \lambda - 1$, are congruent modulo λ to the integers

$$\begin{array}{l} i_1,\ i_1q,\ i_1q^2,\ \cdots,\ i_1q^{t-1}, \\ i_2,\ i_2q,\ i_2q^2,\ \cdots,\ i_2q^{t-1}, \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ i_s,\ i_sq,\ i_sq^2,\ \cdots,\ i_sq^{t-1}, \end{array}$$

in some order, where $s = (\lambda - 1)/t$, and $i_1, i_2, \cdots, i_s$ are the integers less than λ such that the quotient of no two of them is congruent, modulo λ, to a power of q, since this definition of the i's shows that no two elements of the above array are congruent modulo λ. According to the first conclusion regarding factors of $\psi_r(\alpha)$, we may write

$$\psi_r(\alpha) = \prod_a \mathfrak{P}_a^{d_a}, \tag{5}$$

where a ranges over the integers $i_1, i_2, \cdots, i_s$, and d_a is the number of expressions

$$|-ri_aq^{t-j}|+|-i_aq^{t-j}|, \quad (j = 0, 1, 2, \cdots, t-1),$$

whose values exceed λ. Since $\mathfrak{P}(\alpha)=\mathfrak{P}(\alpha^{q^c})$, a an integer, the ideals $\mathfrak{P}(i_nq), \mathfrak{P}(i_nq^2), \cdots, \mathfrak{P}(i_nq^{t-1})$, where $\mathfrak{P}(\alpha^f) = \mathfrak{P}(f)$ are all equal, and the decomposition (5) is identical with the the decomposition (4), which was to be shown. Using Kummer's notation, the relation

$$|-ri_eq^{t-j}|+|-i_eq^{t-j}|>\lambda$$

may be written in the form

$$\gamma_{\mu-h}+\gamma_{\mu-h+\operatorname{ind} r}>\lambda,$$

where γ_b is the least positive integer satisfying $\gamma_b \equiv \gamma^b \pmod{\lambda}$ and ind r is defined by $\gamma^{\operatorname{ind} r} \equiv r \pmod{\lambda}$. In view of the above we may now use the relation*

$$\psi_r(\alpha)^{H_1\lambda} = \pm\, \alpha^d \prod_h F(\alpha^{\gamma^h}) \tag{5a}$$

where $H_1\lambda$ is the class number of $\Omega(\alpha)$, H_1 prime to λ.

Now, as pointed out in my first article, Kummer employs (page 54), without proof or reference the relation

$$\frac{d_0{}^{m\lambda^r}\log\varphi(e^v)}{dv^{m\lambda^r}} \equiv \frac{d_0{}^{m\lambda^r}\log\varphi_1(e^v)}{dv^{m\lambda^r}} \pmod{\lambda^{r+1}},$$

m not being a multiple of $\lambda - 1$, and $\varphi(\alpha) \equiv \varphi_1(\alpha) \pmod{\lambda^{r+1}}$. We shall prove a special case of this relation and use of it will enable us to complete the proof of Theorem IV. Assume that $\varphi(x)$ and $\varphi_1(x)$ are two integral algebraic functions of x with rational integral coefficients, such that $\varphi(1) = \varphi_1(1)$ and $\varphi(\alpha) = \varphi_1(\alpha)$. Further let $\varphi(\alpha)$ be prime to λ, whence it follows that $\varphi(1) \not\equiv 0 \pmod{\lambda}$. Under these conditions, it will be shown that

$$\frac{d_0{}^{m\lambda}\log\varphi(e^v)}{dv^{m\lambda}} \equiv \frac{d_0{}^{m\lambda}\log\varphi_1(e^v)}{dv^{m\lambda}} \pmod{\lambda^2}. \tag{6}$$

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* Kummer gives this relation in a form not containing the factor α^d, but this appears to be an error. Compare the reasoning in the writer's paper, ANNALS OF MATHEMATICS, (2), vol. 21, p. 74, on the determination of $e(\alpha)$ in relation (6). This does not affect Kummer's later arguments, however.

We have

$$\varphi(e^v) = \varphi_1(e^v) + VW_1$$

where $V = (e^{vp} - 1)/(e^v - 1)$ and W_1 is an integral function of e^v. If we divide VW_1 by $e^{vp} - 1$, we have a remainder which must be of the form cV where c is an integer. We then write

$$\varphi(e^v) = \varphi_1(e^v) + W(e^{vp} - 1) + cV.$$

Putting $v = 0$, we have

$$\varphi(1) = \varphi_1(1) + cp,$$

and by hypothesis $\varphi(1) = \varphi_1(1)$, so that $c = 0$. Hence

$$\varphi(e^v) = \varphi_1(e^v) + W(e^{vp} - 1)$$

and

$$\frac{\varphi(e^v)}{\varphi_1(e^v)} = 1 + \frac{W(e^{vp} - 1)}{\varphi_1(e^v)} = U, \text{ say.}$$

We then have

$$(7) \quad \frac{d^{m\lambda} \log \varphi(e^v)}{dv^{m\lambda}} - \frac{d^{m\lambda} \log \varphi_1(e^v)}{dv^{m\lambda}}$$
$$= U^{-1} \frac{d^{m\lambda} U}{dv^{m\lambda}} + (m\lambda - 1) \frac{d^{m\lambda-1} U}{dv^{m\lambda-1}} \cdot \frac{d(U^{-1})}{dv}$$
$$+ \cdots + \frac{dU}{dv} \cdot \frac{d^{m\lambda-1}(U^{-1})}{dv^{m\lambda-1}} \cdot$$

Now every derivative of U is divisible by λ, if zero is substituted for v. Hence every term of the right-hand member of the above relation consists of integers or fractions whose denominators are prime to λ (since $\varphi_1(1) \not\equiv 0 \pmod{\lambda}$) and whose numerators are divisible by λ^2, excepting possibly the first term. To examine this term, we write $X = W/\varphi_1(e^v)$, and, therefore

$$\frac{d^{m\lambda} U}{dv^{m\lambda}} = (e^{vp} - 1) \frac{d^{m\lambda} X}{dv^{m\lambda}} + m\lambda \frac{d(e^{v\lambda} - 1)}{dv} \cdot \frac{d^{m\lambda-1} X}{dv^{m\lambda-1}}$$
$$+ \cdots + X \frac{d^{m\lambda}(e^{v\lambda} - 1)}{dv^{m\lambda}} \cdot$$

Since

$$\frac{d_0{}^k(e^{v\lambda} - 1)}{dv^k} \equiv 0 \pmod{\lambda^2},$$

for $k > 1$, it follows from these relations that

$$\frac{d_0^{m\lambda} U}{dv^{m\lambda}} \equiv 0 \pmod{\lambda^2};$$

and by using this with (7) the congruence (6) is obtained.

From (5*a*) we have

$$\psi_r(\alpha)^{H_1\lambda(\lambda-1)} = \alpha^{d(\lambda-1)} \prod_h F(\alpha^{\gamma^h})^{\lambda-1}, \tag{5b}$$

and we may write $F(\alpha^{\gamma^h})^{\lambda-1} = 1 + \omega(1-\alpha) = G(\alpha^{\gamma^h})$, where ω is an integer in $\Omega(\alpha)$, since if

$$F(\alpha^{\gamma^h}) = a + \omega_1(1-\alpha),$$

where a is a rational integer, then

$$\begin{aligned} F(\alpha^{\gamma^h})^{\lambda-1} &= a^{\lambda-1} + \omega_2(1-\alpha) \\ &= 1 + \lambda k + \omega_2(1-\alpha) \\ &= 1 + \omega(1-\alpha), \end{aligned}$$

after using the known relation $\lambda = \epsilon(1-\alpha)^{\lambda-1}$ where ϵ is a unit in $\Omega(\alpha)$. The relation (5*b*) then becomes

$$\psi_r(\alpha)^{H_1\lambda(\lambda-1)} = \alpha^{d(\lambda-1)} \prod_h G(\alpha^{\gamma^h}), \tag{5c}$$

whence

$$\psi_r(1)^{H_1\lambda(\lambda-1)} = \prod_h G(1) = 1,$$

Since it is known that $\psi_r(1) = 1$. Hence we can apply (6) to (5*c*), and proceeding as Kummer did on pages 54–57 of his article, we obtain

$$\text{ind } E_\nu(\alpha) \equiv -\frac{(-1)^{\lambda+\mu}(\gamma^{2\nu}-1)B_{\nu\lambda-\mu}}{2H_1\lambda} \cdot \frac{d_0^{\lambda-2\nu} lG(e^v)}{dv^{\lambda-2\nu}},$$

modulo λ. By definition of G we also have

$$\frac{d_0^{\lambda-2\nu} lG(e^v)}{dv^{\lambda-2\nu}} \equiv -\frac{d_0^{\lambda-2\nu} lF(e^v)}{dv^{\lambda-2\nu}} \pmod{\lambda},$$

and the last two relations give the congruence obtained by Kummer (loc. cit., page 57). The remainder of his argument, up to page 61, appears to be correct, if we use the Dedekind definition of ideal. This establishes the theorem which I have designated IV.

CORNELL UNIVERSITY

HAHN'S REELLE FUNKTIONEN

Theorie der reellen Funktionen. By Hans Hahn. Vol. I. Berlin, Julius Springer, 1921. vii + 600 pp.

A preliminary indication of the contents of a book may sometimes be conveyed by a statement as to what it does not contain. Such information is supplied by the preface to the present volume, which states that a second volume, completing the work, is to present the theory of integration and differentiation, the analytic representation of arbitrary functions, and Fourier's series. A lower inequality for the order of ideas involved is given by another statement in the preface, that although the principal facts of the general theory of sets and the theory of real numbers are summarized in an introduction for convenience of reference, no systematic development of these theories is attempted, and the difficult questions which gather around their foundations are not touched upon at all.

The prospective reader will be further enlightened by a glance at the author-index at the back of the book. There he will find thirty-eight references to Lebesgue, thirty-three to W. H. Young, and twenty-four each to Baire and Hausdorff, while, at the other end of the scale, Heine is mentioned four times, Cauchy three times, Dirichlet twice, and Riemann once.

The volume is concerned, then, with those investigations of the last few decades which have had the specific purpose of throwing the fullest and most searching light on the fundamental concepts of function and limit. In a larger sense, the working out of these concepts may be regarded as the supreme achievement of research in mathematical analysis during the past ninety or a hundred years. The various special theories, while offering more or less that is of value in their own particular conclusions, have derived an added significance from their relation to the central theme, the bringing of the mysteries of eighteenth-century and early nineteenth-century mathematics within the domain of assured knowledge and common-sense. Our files of journals doubtless contain thousands of pages which will receive scant attention from generations to come, but if this generalization of an existence theorem, or that simplification of a convergence proof, has contributed in its day to the stirring of ideas through which understanding emerges, it has done its part. From this point of view, the present work represents a somewhat comprehensive version, rather than a fragment, of what has been learned in a century of analysis. It is a summary too general in its terms, too abstract and refined, to be appreciated by limited human intelligence without a background of experience and illustration, which the reader must bring to its study; but it is an account possessing a certain symmetry and completeness of its own.

It goes without saying that a book which opens with the words "We begin with a brief survey of the simplest facts in the theory of sets," which presents the Wohlordnungssatz on page 25, and which in six hundred pages does not reach the subject of integration or differentiation, is not designed as an introductory text. An adequate appraisal of it as a work of science would require the insight of a critic much more widely and deeply read in the literature of the field than the present reviewer. His naïve impressions on turning its pages may, however, be of some value to other readers of similarly unspecialized qualifications.

As has been mentioned already, an introductory chapter summarizes the facts that are to be used subsequently with regard to sets in general, the transfinite cardinal and ordinal numbers, the real number system, and the various types of limits associated with sequences and sets of real numbers. There is no attempt to lay down a complete system of definitions and postulates. There is no definition of an infinite set, for example, and an irrational number is defined merely as a number that is not rational. The principle of choice is accepted with a simple statement that criticism of its logical basis need not be taken into account for the purpose in hand. It is used, or Zermelo's theorem based on it is used, without further question, when occasion demands, and at least once when it is not necessary, in the proof of the theorem that two sets are equivalent if each is equivalent to a part of the other. Nevertheless, one must not take too seriously the author's modest statement that the treatment is not "systematic." It will be an unusual reader whose knowledge of the subject is not materially improved both in range and in precision by a perusal of the chapter.

For the purposes of a study which is ultimately to be concerned largely with derivatives and derived numbers, it is convenient to permit functions to take on infinite values, and the real number system is formally extended in the present treatment by the adjunction of the two numbers $+\infty$ and $-\infty$. The careful working out of the idea shows clearly the difficulties with which any such definition has to contend. It is a matter of choice whether infinity shall be regarded as a number or not, but no definition can make it a number *like other numbers*. Nearly a page is devoted to an enumeration of the operations which may and those which may not be performed with the two infinite numbers. But complications are not thereby finally disposed of. For example, the operation $(+\infty) - (+\infty)$ is in general not defined (p. 28). But when the oscillation (Schwankung) of a function is defined, in a later chapter (pp. 190–191), it is stated that, for this particular purpose, $(+\infty) - (+\infty)$ shall be regarded as having the value 0. And yet, in a subsequent discussion of the same concept (pp. 214–215), it is found necessary to regard infinite values as exceptional again.

The limit of a sequence of numbers is so defined as to admit the possibility of infinite limits. "A sequence which has a limit is called convergent; in particular, if the limit is finite, the sequence is said to be properly convergent; a sequence that does not converge is called oscillating" (p. 32). The same convention is maintained in the definition of continuity (pp. 122–123), and even in those of uniform continuity (p. 131) and uniform convergence (pp. 246–247). On the other hand, when the limit of a sequence of points is introduced, while the definition that recognizes infinite limits is mentioned, it is not the one adopted for systematic use (pp. 56–57). The sequence of numbers 1, 2, 3, . . . , has the limit $+\infty$, while the corresponding sequence of points on the axis of reals (with the customary definition of distance) has no limit.

In calling attention to these contrasts, the reviewer does not wish to deprecate the introduction of infinite numbers. On the contrary, he regards the author's careful treatment of them, in a form for convenient reference, as a particularly valuable service to students of analysis. But it must be recognized that their use is at best the choice of the lesser of two considerable evils. The difficulties are in the nature of things, and are not lightly to be smoothed away.

Chapter I, on point sets, opens with the postulates for distance in a general metric space. The rest of the treatise is similarly general in scope, with occasional specific consideration of relations in euclidean space of one or more dimensions. This does not mean, however, that its aspect is uniformly forbidding to the general reader. On the contrary, it is singularly easy to dip in almost anywhere and gain at least a partial appreciation of what is going on. For example, one reads on page 86: "A part $\mathfrak{A}'$ of an arbitrary set $\mathfrak{A}$ is called a component of $\mathfrak{A}$, if $\mathfrak{A}'$ is connected, and every connected part of $\mathfrak{A}$ which has a point in common with $\mathfrak{A}'$ is a part of $\mathfrak{A}'$." The general intent of this is fairly obvious at a glance, although a precise understanding of it naturally involves a knowledge of the technical meaning of the terms involved; and if the reader turns back a few pages and finds that "A set $\mathfrak{A}$ is said to be connected, if it is not the sum of two non-vacuous parts, each closed on $\mathfrak{A}$," he may come to the conclusion that there is more in the definition first quoted than he thought. Of course it occasionally happens that one misses the point of a statement entirely, unless one has the exact meaning of the terminology in mind, as, in the latter definition, the distinction between *sum* (Summe) and *union* (Vereinigung) of two sets (p. 2). For the most part, however, be it repeated, a fair understanding of the text is surprisingly accessible even to the casual reader, such is the excellence of the presentation.

Chapter II deals with the concept of function, upper and lower bounds and limits of a function on a point set or at a point, continuity, and semicontinuity. It is to be remembered that the terms point and point set refer to elements and sets of elements in any kind of "space" which satisfies the necessary postulates. Mention has already been made of the admission of infinite values for a function; the definitions are of course framed for functions defined on an arbitrary point set, so that, for example, "at an isolated point of $\mathfrak{A}$ every function f is continuous on $\mathfrak{A}$" (p. 125), whether its value there is finite or infinite. Attention may be called to the sections on the continuous extension of a continuous function beyond its original domain of definition, and on correspondences between point sets, such correspondence being treated as a generalization of the function idea.

Chapter III is concerned with discontinuous functions, the oscillation functions associated with them, the distribution of points of discontinuity, and pointwise discontinuous functions in particular. The next chapter treats of sequences of functions: of continuous convergence at a point, uniform convergence at a point or over a point set, and related concepts. The fifth chapter gives a very systematic and satisfying account of the

Baire classification of functions. The last three chapters, on absolutely additive functions of sets, functions of limited variation, and measurable functions, cover a considerable part of what is commonly regarded as the theory of integration, though the process of integration itself is not discussed. The treatment of the measure of point sets in the sixth chapter is an especially interesting example of the method of postulates, several groups of theorems being developed in succession as successive restrictions are placed on the notion of measure.

The material appearance of the book is indicative of unremitting effort under adverse conditions. The paper, though fairly white and clear, is almost of the consistency of blotting-paper; the typography, on the other hand, is so excellent that a fairly attentive turning of the pages has disclosed only some half-dozen trifling misprints.

The exposition is remarkably clear and systematic throughout. If sometimes rather diffuse for a mere presentation of the facts, it is the more convenient for purposes of reference. The book is written neither as an exhibition of the author's learning, nor as a memorial to an abstractly conceived body of truth, orderly and symmetric as it is; it is written for the reader's information, by a man who anticipates the reader's difficulties and provides against them with great faithfulness and skill. If it be true that genius is nothing more than an infinite capacity for taking pains, this is unquestionably a work of genius; if it is possible to imagine a certain quality of inspiration which seldom thrills the reader of these pages, it is perhaps only because passages which would be regarded as evidence of inspiration in the case of a less eminently qualified writer are here accepted as a mere matter of course.

DUNHAM JACKSON.

SHORTER NOTICES

The Mathematical Theory of Electricity and Magnetism. By J. H. Jeans. Fourth Edition. Cambridge, University Press, 1920. vi + 627 pp.

Jeans's text, first printed while he was at Princeton fifteen years ago, has ever since been one of the very few standard texts in its particular field. Some substantial rearrangements and additions were made in the second edition (1911), and more are now found in the fourth, especially with reference to relativity. Nothing is offered on quantum theory or on the modern electrical theory of matter. Though there is some account of electron theory (of the Lorentz-Drude type), the book is essentially an introduction to the Maxwell theory with the additions of those parts of electromagnetic theory most naturally flowing out of Maxwell's point of view. The reason might not be far to seek: the work is, as it states, on the mathematical theory of electricity and magnetism, written by one who revels in the mathematical treatment of nature's problems, and the most recent developments of physical hypothesis relative to the electrical and quantum theories of matter are hardly yet emerging into a mathematical theory of sufficient completeness and elegance to rejoice the heart.

I have read Jeans's book more than once with classes of students who had had a good training in calculus and in the elements of electricity and magnetism; we have found in the text and problems excellent and practicable material for a full year's course with plenty to spare. It is good for young people who plan to be physicists or higher engineers to study the text and to work the problems. A facility is thereby gained in the application of analytic processes to nature. Better headway could be made if students had a better training in analytic mechanics. Greater interest could be more readily maintained if the book were more physical, if more problems were stated in terms of volts, ampères, ohms, etc., and if a greater emphasis were laid on checking the results against experiment. A student may do excellent work upon this text without gaining at all in his sense of contact with nature,—this is often the case with mathematical physics.

Pedagogically the text is very uneven, now easy, now difficult; some of the exercises are almost trivial, others exceed the ability of the best students. The teacher must be wary. Inasmuch as we are offered ample material for two years of study and in view of the real teaching difficulty of developing in a student at one time both the physical and the analytic sense, I incline to the belief that the text would be greatly improved by a major rearrangement of the material, including the exercises, so that first one might have the physical developments and those exercises in which the difficulties are the physical statement and the analysis of the problem, and second the mathematical elaboration and the exercises in which the tricks of mathematical analysis offer the obstacles to be overcome. It is not easy for the teacher to work out this assortment from the text as now composed. However, I do not know whither I should turn for a book that would work out any better for my course on the mathematical theory of electricity and magnetism.

E. B. Wilson.

Problems of Cosmogony and Stellar Dynamics. By J. H. Jeans. Adams Prize Essay for 1917. Cambridge, University Press, 1919. vii + 293 pp.

As a general cultural study, astronomy is comparable to poetry in its stimulation of the imagination. It appeals to the widest interests, the most diverse individuals. And of all astronomy the most fascinating is cosmogony—the origin, evolution, and destiny of our material universe. The arrangement pursued by Jeans in his Adams Prize Essay was apparently adopted with the aim of permitting an intelligent reader to carry the general drift of his theory without having to wade through deterring mathematical difficulties. Recondite mathematics there is in plenty, but always with a pleasant prelude and with an effective summary.

Briefly, Jeans's conception of stellar evolution follows Russell and his notion of the cosmos as a whole is the "island universe" theory. Our galactic system is comparable in mass with the largest spiral nebula. The spiral nebulæ are not part of our galactic system, which is a star cluster larger than other star clusters as the Andromeda nebula is larger than other spirals. The development of the individual stars is from cold diffuse masses of cosmic dust up through the red to the white or even blue type, and then back to the dense dying reds. The theories of figures of equilibrium changing slowly toward a limiting unstable state with subsequent rapid cataclysmic development are marshalled in support of this general conception of the world. That some differ with the author even in the dynamical theory, to say nothing of the general conclusions, may be seen by reference to Macmillan's long analytical review in the ASTROPHYSICAL JOURNAL. For a discussion of the tenability of the island-universe theory, one may read the symposium of Curtis and Shapley before the National Academy in 1920, printed by the National Research Council.

As I am merely one of the many who have read avidly the writings of mathematical and physical astronomers since the time I could first read at all, I have but small right to any opinion on cosmogony, but may I venture a word in admiration of the courage of those who work in this difficult field where there are rare clews and, it sometimes seems, no proofs. There are only seven stars (of course, binaries) of which the masses are well determined, and these lie between 0.7 and 3.4 if the sun's mass be unity; yet it is generally stated that the masses of all the stars are of approximately the same magnitude. This is a courageous conclusion. May it not be that the eight (our sun included) are not a fair random sample of the billion and a half? And even if you have a random sample of eight out of a lot of a billion and a half what is the probable departure of the average of any particular physical property of the eight from that of the lot? Jeans is aware of such troubles; he is tentative, not dogmatic. (Since this review was first written, a discovery has been reported of a double star system of which the mass is many times that of the sun.)

The essay contains a review of earlier work by Roche, Darwin, Poincaré, expanded with many individual contributions which the author has been making in recent years in friendly competition with Eddington. Those who read the work will feel the thrill of Keats "On First Looking into Chapman's Homer."

E. B. WILSON.

Storia della Geometria Descrittiva dalle Origini sino ai Giorni Nostri. By Gino Loria. Milano, Ulrico Hoepli, 1921. xxiv + 584 pp.

Descriptive geometry has not been neglected in the histories of mathematics and considerable historical information is to be found in some of the text-books. We have also Poudra's *Histoire de la Perspective Ancienne et Moderne* (1864), Cremona's *Sulla Storia della Prospettiva Antica e Moderna* (1865), and Obenrauch's *Geschichte der darstellenden und projectiven Geometrie* (1897). Nevertheless, there is a place for a book of the present day devoted exclusively to the evolution of descriptive geometry,—especially when it is written by one so well qualified for the task as is Professor Loria.

As the author considers descriptive geometry a branch of pure mathematics, he treats briefly the history before the time of Monge. This is done in the first three chapters, which contain one-sixth of the total number of pages of the text. The next two chapters, which add almost another sixth to the book, deal with Monge and his contemporaries and disciples. The treatment of the material in the first hundred and twenty pages of the text is not new to the reader who is familiar with Professor Loria's article (XXV) in the fourth volume of Cantor's *Vorlesungen über Geschichte der Mathematik.*

The further development of descriptive geometry until about 1885 occupies seven chapters, almost one-half of the text. Six of these follow the history in different countries. In the chapters on Italy, France, Germany, German Switzerland, and Austria-Hungary, a short preliminary statement of conditions is followed by separate sections devoted to critical studies of the writings of the principal men. These chapters are concluded by sections treating the less important writers. The other countries are disposed of in one short chapter. The twelfth chapter deals with a special subject, axonometry, and traces its development from 1820 to the present.

In the last chapter of the book (the development from 1885 onward) the separate sections are entitled: (1) general considerations, (2) old and new methods of representation, (3) plane and space curves, (4) surfaces, (5) illumination of surfaces, (6) descriptive geometry of ruled, of higher, and of non-euclidean spaces, and (7) teaching and history of the subject. In a three-page conclusion the author sets forth the lines along which he predicts that the descriptive geometry of the future can and will develop.

The twelve-page index of proper names is a valuable addition to the text, although there are some misprints. It contains about seven hundred fifty names and one hundred thirty references to Monge. In footnotes scattered throughout the text there are more than a thousand references.

The author has gathered together a large collection of facts. He has arranged them methodically and he has given his sources. And he has done something more, for he has shown an appreciation of relative values by stressing the more important things and subordinating the minor details, he has displayed an insight into the vital relation between his own subject and other branches of learning, and he has created an atmosphere that is likely to inspire the reader with his own enthusiasm for descriptive geometry.

E. B. Cowley.

Plane Algebraic Curves. By Harold Hilton, M.A., D.Sc. Oxford, The Clarendon Press, 1920. 16 + 388 pp.

This is the first treatise on plane algebraic curves to appear in English since Salmon's famous treatise which was published over forty years ago. During the last four decades many new theorems have appeared in the various mathematical journals and thus it is fitting that a new English treatise should be written.

In dealing with such an extensive subject it is of course impossible to include all the known material, and one must expect to find omissions of certain topics. The most striking omission in this text is the modern algebraic-geometric development as set forth by Castelnuovo, Severi, Segre and others. In fact, the whole topic of geometric transformations and the derivation of properties of algebraic curves by means of these transformations has been omitted except for two cases of the quadratic transformations. Thus one is impressed by the fact that this treatise follows to a great extent the same general course as Salmon's, except that topics are more fully discussed and are brought up to date. However, many excellent collections of exercises are scattered throughout the book, and these alone are well worth the price of the text. The great majority of the results have been derived by algebraic methods. In many places the work could be shortened if synthetic methods were used.

The main topics discussed are: coordinate systems, projection, singular points, curve tracing, tangential equations and polar reciprocals, foci, superlinear branches, polar curves, Hessian, Steinerian, Cayleyan, Jacobian of three curves, Plücker's numbers, deficiency, higher singularities, two types of quadratic transformations, unicursal curves, derived curves, intersection of curves, unicursal cubics, non-singular cubics, cubics as Jacobians, use of parameter for non-singular cubics, unicursal quartics, quartics of deficiency one and two, non-singular quartics, ovals and circuits, corresponding ranges and pencils.

This book should be found in every mathematical library, for the topics discussed are most admirably treated. As a text for a first course it is superior to anything that has appeared as yet in any language because of the excellent collection of exercises.

F. M. Morgan.

Vector Calculus. By Durgaprasanna Bhattacharyya. Calcutta, University of Calcutta, 1920. 90 pp.

In order to present the essential features of vector analysis for use in mathematical physics, the author develops the differential and integral calculus of vectors and functions of vectors directly. He scorns the use of coordinates very properly with much the feeling of Tait that one "should not violate the spirit of the Order." However, he does not overlook the great practical advantage that accrues from the study of ordinary vectors.

The author defines line, surface, and volume integrals of vectors and of vector functions. He then defines gradient of a scalar function F substantially as the vector F in the expression $d\rho \cdot F$. After considering the linear vector function of a vector, he proceeds to the differential calculus of vectors. He uses essentially the derivative dyad $d\sigma = d\rho\cdot\sigma$, that is, the dyad σ. It is now easy to define divergence as the average value of the scalar $\alpha\cdot\sigma\cdot\alpha$ over the surface of a unit sphere, α being any unit vector of variable direction. This gives of course what is commonly written $\cdot\sigma$. In the same manner the average value of $\alpha \times \varphi\alpha$ (where $\varphi\alpha = \alpha\cdot\sigma$) over the surface of the unit sphere, gives the curl. This would be the same as $\times\,\sigma$. There is nothing new in these definitions, as they have been given in one form or another before. A development of formulas that are useful follows.

Part II of 33 pages discusses the steady motion of a solid under no forces in liquid extending to infinity. Some of the well known results are reached. Towards the latter part of the section he considers certain cases of stable motion. The general problem is rather intractable, and the author is content with stating some conclusions of the simplest case. He finds that for this simple case, the two steady motions for which the screws are parallel to the greatest and least axes of a certain ellipsoid, are stable; that steady motion for which the screw is parallel to the mean axis is unstable.

J. B. SHAW.

An Introduction to the Theory of Relativity. By L. Bolton. New York, E. P. Dutton and Company, 1921. ix + 177 pp.

This little book gives a remarkably readable and intelligible account of the theory of relativity. It is by the author of the prize-winning essay in the contest recently conducted by the SCIENTIFIC AMERICAN. The author admits at the outset the impossibility of giving any sort of adequate notion of the theory of relativity without the use of mathematical ideas and symbols. He has set himself the task, however, of presenting his material without presupposing more than elementary algebra and the elements of plane geometry. As a result he finds it necessary to introduce the reader to the notion of the differential of arc, and in so doing brings back to life the "little zeros." There is also (p. 126) a footnote implying that cones and cylinders are the only developable surfaces. Such incidental blemishes may, however, be excused in view of the book's purpose and the extraordinarily satisfactory way in which this purpose has been carried out.

J. W. YOUNG.

Geometria Analytica. Part 1. *Il Metodo delle Coordinate.* By L. Berzolari. Second edition. Milan, Ulrico Hoepli. xiii + 495 pages.

This work belongs to the excellent series of manuals of the firm of Ulrico Hoepli. It aims to define the principal systems of coordinates in space of one, two and three dimensions and to derive the principal theorems and formulas of analytic geometry connected with them. It discusses in detail Cartesian coordinates of points, of lines in a plane and of planes in space. It further deals with homogeneous coordinates, and the fundamentals of analytic projective geometry. Equations of curves and surfaces are properly treated briefly. A short but well-selected appendix on vector analysis is added to this edition. Such topics as line coordinates in space and coordinates of spheres do not fall within the scope of this work. A few omitted topics which ought to have been included are: polar coordinates in space (radius vector and direction cosines), cylindrical coordinates and the fundamentals of intrinsic geometry.

The text is well and carefully written and, within its field, is thorough. Pedagogical requirements are borne constantly in mind and the expository form is good. American teachers and authors will find it a useful work for reference.

CHARLES SISAM.

Essai philosophique sur les Probabilités. By Pierre-Simon Laplace. I, II. Paris, Gauthier-Villars, 1921. 12 + 104 + 108 pp.

Mémoire sur la Chaleur. By MM. Lavoisier et de Laplace. Paris, Gauthier-Villars, 1920. 78 pp.

Mémoires sur l'Électromagnétisme et l'Électrodynamique. By André-Marie Ampère. Paris, Gauthier-Villars, 1921. 14 + 112 pp.

The editor of this series of reprints, entitled *Les Maîtres de la Pensée Scientifique*, pertinently remarks that the rapid scientific advances of the present time tempt us to neglect past discoveries and their authors. This neglect is almost unavoidable whenever the original papers are not within reach of the mass of scientific students. Reprints in cheap form, like the above, should meet with a hearty welcome. It is a privilege to be able to carry around in one's coat pocket the masterpieces of Laplace, Lavoisier and Ampère. Surely there is no need of enlarging upon the commanding place which each of these memoirs occupies in the history of science.

FLORIAN CAJORI.

Gruppentheorie. By Dr. L. Baumgartner. Berlin, Walter de Gruyter & Co., 1921. 120 pp.

This little book is as interesting as it is handy. It is simply written and divided into sections in a helpful way. There are four chapters: Introduction to the Group Notion, The Group Notion in Geometry, The Finite Group, The Infinite Group. The third chapter is much the longest and perhaps the most unified.

There are many illustrative examples, from theory of functions, transformations, number theory, etc., the predominance being from geometry. The literature list (p. 5) is very brief; in English only Burnside's earlier book is mentioned. The name of Lagrange is applied to the theorem, The order of a subgroup of a finite group is a factor of the order of its group (p. 7), a misnomer we are now told.* Sylow's theorem is introduced (p. 98) but a proof is appropriately omitted. If one were using this book for a first study of groups, certainly many supplementary exercises would be desirable. The little volume closes with answers to its forty-nine exercises and a very useful index.

LOUIS C. MATHEWSON.

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* Carmichael, R. D., this BULLETIN, (2), vol. 27 (1922), pp. 474, 475.

NOTES

The opening number of volume 23 (1921–22) of the ANNALS OF MATHEMATICS contains the following papers: *On matrices whose elements are integers*, by Oswald Veblen and Philip Franklin; *An algorism for differential invariant theory*, by O. E. Glenn; *The general theory of cyclic-harmonic curves*, by R. E. Moritz; *More theorems on the complete quadrilateral*, by J. W. Clawson; *A theorem on cross-ratios in the geometry of inversion*, by J. L. Walsh; *The condition for an isothermal family on a surface*, by J. K. Whittemore; *The reversion of class number relations and the total representation of integers as sums of squares or triangular numbers*, by E. T. Bell; *Note on the term maximal subgroup*, by G. A. Miller; *Reducible cubic forms expressible rationally as determinants*, by L. E. Dickson; *Note on the Picard method of successive approximations*, by Dunham Jackson; *A fundamental system of covariants of the ternary cubic form*, by L. E. Dickson; *The modular theory of polyadic numbers*, by A. A. Bennett; *Some algebraic analogies in matric theory*, by A. A. Bennett; *Generalized conjugate matrices*, by Philip Franklin.

A Benjamin Pierce Instructorship in Mathematics at Harvard University (see this BULLETIN, vol. 21, page 315) is open to general competition. Applications for the year 1923–24, accompanied by the necessary papers, should reach Professor Birkhoff not later than February 15, 1923, and all communications relating to this appointment should be addressed to him.

Max Gutzmer, of Berlin, has presented the Leopoldinisch-Carolinische Akademie der Naturforscher with a sum of money as an endowment of a prize for important German methematical work. Details concerning the conditions of award will be announced later by the Academy.

Professor A. Wangerin has been elected honorary member of the Leopoldinisch-Carolinische Akademie, and has been awarded its Cothenius Medal for his mathematical work.

The Accademia dei Lincei has awarded its royal prize for astronomy for the period 1915–1920 to Professor G. Armellini, of the University of Pisa.

Dr. A. Fraenkel has been promoted to an associate professorship at the University of Marburg.

Dr. T. Kaluza has been promoted to an associate professorship at the University of Konigsberg.

Professor R. König, of the University of Tübingen, has been appointed professor of mathematics at the University of Münster.

Dr. H. Rademacher, of the University of Berlin, has been appointed to an associate professorship at the University of Hamburg.

Professor L. Hotopp, of the Hannover Technical School, has retired from active teaching.

Dr. E. Feyer has been admitted as privatdocent at the Breslau Technical School.

At the University of Geneva, Dr. D. Mirimanoff has been appointed associate professor of the calculus of probabilities, and Dr. R. Wavre associate professor of the calculus and rational mechanics.

Professor H. Beghin, of the Ecole Navale, and Dr. Soula have been appointed maîtres de conférences in mathematics at the University of Montpellier.

The following have been admitted as privatdocents at Italian universities: G. Andreoli, at the University of Naples; G. Aprile, at the University of Catania; G. Mignosi, at the University of Palermo; R. Serini and G. Usai, at the University of Pavia.

Mr. R. A. Sheets, of Denison University, has been promoted to an assistant professorship of mathematics.

Miss Evelyn Walker has been promoted to an assistant professorship of mathematics at Hunter College.

Associate Professor Gillie A. Larew, of Randolph-Macon Woman's College, has been promoted to a full professorship of mathematics.

Associate Professor H. H. Conwell, of Beloit College, has been promoted to a full professorship of mathematics.

Mr. David Hiches has been appointed professor of mathematics at Creighton University.

At the University of Wyoming, Professor C. E. Stromquist is on leave of absence on account of illness. Professor H. C. Gossard, who has for two years served as regional secretary of the Student Y. M. C. A., has returned to teaching and is acting head of the department of mathematics.

At Amherst College, Professor E. L. Dodd, on leave of absence from the University of Texas, is taking the place of Professor J. G. Hardy.

At Washington University, Mr. Theodore Doll of Northwestern University has been appointed assistant professor of mathematics.

At the University of Virginia, Mr. F. L. Brown, of Northwestern University, has been appointed assistant professor of mathematics.

At the University of Iowa, Dr. W. H. Wilson has been appointed assistant professor of mathematics, and Dr. Roscoe Woods has been appointed associate in mathematics.

The following appointments to instructorships in mathematics in American colleges and universities are announced:

University of Akron, Miss W. H. Lipscombe;
Albion College, Mr. Leon Sears;
Beloit College, Mr. R. M. Robinson;
University of California, Dr. Elsie McFarland; Southern Branch, Dr. F. C. Leonard;
University of Chicago, Dr. Mayme I. Logsdon;
Colgate University, Mr. H. A. D. Bell;
Cornell College, Mount Vernon, Mr. Edmund Ingalls;
Denison University, Mr. H. B. Lemon;
Georgia School of Technology, Mr. G. T. Trawich;
University of Georgia, Mr. A. H. Stevens;
Kenyon College, Mr. F. L. White;
Lake Forest College, Miss Marie M. Johnson;
Lehigh University, Mr. H. S. Bunn;
University of Maine, Mr. F. S. Beale, Mr. Edward Brow, Mr. W. E. Loring;
Northwestern University, Mr. R. L. Jackson, Mr. J. D. Voss;
Ottawa University, Mr. H. W. Bailey;
University of North Dakota, Mr. J. D. Leith;
University of Pittsburgh, Mr. Z. N. Holler;
Princeton University, Dr. Einar Hille, Mr. G. B. Briggs;
Sweet Briar College, Miss Mary Searle;
University of Texas, Mr. J. E. Burnam, Mr. R. Lubben;
Tufts College, Mr. P. D. Wilkins;
Wheaton College, Miss Martha F. Chadbourne;
University of Wisconsin, Miss Camilla Hayden.

Professor A. Höfler, of the University of Vienna, died February 26, 1922, at the age of sixty-eight years.

Professor E. Kötter, of the Technical School at Aachen, died January 26, 1922, at the age of sixty-two years.

Professor C. Kostka, of Insterburg, died December 28, 1921, at the age of seventy-five years.

Professor Axel Thue, of the University of Christiania, died March 7, 1922, at the age of fifty-nine years.

Professor Ernest Lebon, of the Lycée Charlemagne, Paris, died February 12, 1922, in his seventy-sixth year.

Dr. A. R. Willis, of the department of mathematical physics at the Royal College of Science, died June 23, 1922, at the age of seventy-two years.

Rear Admiral C. H. Davis, U. S. N., died at Washington December 27, 1921, at the age of seventy-six years. He was twice superintendent of the Naval Observatory.

Emeritus Professor A. A. Markov, of the University of Petrograd, died July 27, 1922, at the age of sixty-six years.

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CONDITION THAT A TENSOR BE THE CURL OF A VECTOR*

BY L. P. EISENHART

It is the purpose of this note to establish the following theorem.

THEOREM. *A necessary and sufficient condition that a covariant skew-symmetric tensor* A_{ij} *in a space of any order n be expressible in terms of n functions* φ_i *in the form*

$$A_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i} \tag{1}$$

is that

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0, \quad (i, j, k = 1, \cdots, n). \tag{2}$$

Consider first the case of 3-space. If φ_2 and φ_3 are any two functions such that

$$A_{23} = \frac{\partial \varphi_2}{\partial x^3} - \frac{\partial \varphi_3}{\partial x^2},$$

the conditions of integrability of

$$\frac{\partial \varphi_1}{\partial x^2} = \frac{\partial \varphi_2}{\partial x^1} + A_{12}, \qquad \frac{\partial \varphi_1}{\partial x^3} = \frac{\partial \varphi_3}{\partial x^1} + A_{13}$$

are satisfied in consequence of (2), and the theorem is established for 3-space.

Now we show that, if the theorem is true for n-space, it is true for $(n+1)$-space. On this assumption equations (1) hold for $i, j = 1, \cdots, n$. For a particular i and j and for $k = n+1$, equation (2) may be written in the form

$$\frac{\partial}{\partial x^i}\left(A_{jn+1} - \frac{\partial \varphi_j}{\partial x^{n+1}}\right) = \frac{\partial}{\partial x^j}\left(A_{in+1} - \frac{\partial \varphi_i}{\partial x^{n+1}}\right).$$

Hence a function φ_{n+1} is defined by the equations

$$A_{in+1} = \frac{\partial \varphi_i}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^i}, \qquad A_{jn+1} = \frac{\partial \varphi_j}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^j}. \tag{3}$$

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* Presented to the Society, September 7, 1922.

Replacing j in (2) by $l\ (= 1, \cdots, n;\ \neq j)$, we have, by (3),

$$\frac{\partial A_{l\,n+1}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{\partial \varphi_l}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^l}\right).$$

Consequently (1) holds for $i, j = 1, \cdots, n+1$, and the theorem is established. It should be remarked that one of the functions φ_i may be chosen arbitrarily, or what is equivalent, that the functions φ_i are determined to within additive functions $\partial\psi/\partial x^i$, where ψ is an arbitrary function of the x's.

Thus far we have made no use of the fact that A_{ij} are the components of a covariant tensor. If $A'_{\alpha\beta}$ denote the components of the tensor in terms of coordinates x', then

$$A'_{\alpha\beta} = A_{ij}\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}. \tag{4}$$

If Γ^i_{jk} and $\Gamma'^\alpha_{\beta\gamma}$ denote the Christoffel symbols of the second kind for the respective systems of coordinates x and x' of a Riemannian geometry, then[*]

$$\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} = \Gamma'^t_{ij}\frac{\partial x^p}{\partial x'^t} - \Gamma^p_{qr}\frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j}.$$

The same equations obtain in the more general case of a geometry of paths, where the functions $\Gamma^i_{\alpha\beta}$ and $\Gamma'^\alpha_{\beta\gamma}$ are the coefficients of the equations of the paths in the two systems of coordinates.[†] By means of these equations we show that, if the functions A_{ij} satisfy (2), so also do $A'_{\beta\gamma}$ defined by (4). In consequence of the above theorem equation (4) may be replaced by the equation

$$\begin{aligned}\frac{\partial \varphi_\alpha'}{\partial x'^\beta} - \frac{\partial \varphi_\beta'}{\partial x'^\alpha} &= \left(\frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}\right)\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta} = \frac{\partial \varphi_i}{\partial x'^\beta}\frac{\partial x^i}{\partial x'^\alpha} - \frac{\partial \varphi_j}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}\\ &= \frac{\partial}{\partial x'^\beta}\left(\varphi_i\frac{\partial x^i}{\partial x'^\alpha}\right) - \frac{\partial}{\partial x'^\alpha}\left(\varphi_j\frac{\partial x^j}{\partial x'^\beta}\right).\end{aligned}$$

Hence

$$\varphi_\alpha' = \varphi_i\frac{\partial x^i}{\partial x'^\alpha} + \frac{\partial \psi}{\partial x'^\alpha}, \tag{5}$$

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[*] Bianchi, *Lezioni*, vol. 1, p. 64.

[†] See PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 21.

where ψ is an arbitrary function.

From (5) it is evident that if A_{ij} are defined as the components of the curl of covariant vector, then (2) are necessarily satisfied; but (2) is not a sufficient condition. That this condition is not sufficient was overlooked by me in a recent paper,* and my conclusions in § 5 are not correct. In fact, the skew-symmetric tensor there defined by S_{ij} is given by

$$S_{ij} = \frac{\partial \Gamma^{\alpha}_{\alpha j}}{\partial x^{i}} - \frac{\partial \Gamma^{\alpha}_{\alpha i}}{\partial x^{j}},$$

and the functions $\Gamma^{\alpha}_{\alpha i}$ and $\Gamma'^{\alpha}_{\alpha i}$ in two sets of coordinates are in the relation

$$\Gamma'^{\alpha}_{\alpha i} = \Gamma^{\alpha}_{\alpha j} \frac{\partial x^{j}}{\partial x'^{i}} + \frac{\partial}{\partial x'^{i}} \log \Delta,$$

where Δ is the Jacobian of the transformation.

PRINCETON UNIVERSITY

A NEW GENERALIZATION OF TCHEBYCHEFF'S STATISTICAL INEQUALITY

BY B. H. CAMP

1. *Introduction.* If $f(x)$ is any frequency distribution, and s its standard deviation, the symbol $P(\lambda s)$ may be used to represent the probability that a datum drawn from this distribution will differ from the mean value by as much as λs, numerically. For the solution of various statistical problems it is desirable to have a formula which will measure $P(\lambda s)$ when $f(x)$ is only partially known. A case of practical importance occurs when $f(x)$ represents the distribution of values of a statistical constant determined by sampling from a known distribution, such a constant as, for example, a mean value, or a coefficient of correlation. In such cases it is usually difficult or impossible to find the complete distribution $f(x)$, but quite feasible to find its lower moments. Tchebycheff's well known inequality is: $P(\lambda s) \leqq 1/\lambda^2$. It has been general-

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* PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 236.

ized in a formula first discovered by K. Pearson:*

$$P(\lambda s) \leqq \frac{\beta_{2r-2}}{\lambda^{2r}}, \qquad (r = 1, 2, \cdots),$$

where $\beta_{2r-2} = m_{2r}/s^{2r}$, m_{2r} being the $2r$th moment of $f(x)$ about its mean. Pearson's view is that, although in most cases this is a closer inequality than Tchebycheff's, it is usually not close enough to be of practical assistance. However, it is the best formula so far obtained which logically can be used with distributions whose larger even β's depart considerably from their Gaussian values. It is proposed to exhibit here a method by which the right side of Pearson's inequality may usually be decreased by about fifty per cent. The theory will be explained in § 2, and illustrated in § 3.

2. *Theory.* LEMMA. *Let* $Q(t) \geqq 0$, *and be monotonic, decreasing, and let* $d^2Q/dt^2 \geqq 0$, *in the interval* $1 \leqq t \leqq k$. *Let the straight line* $y = a + bt$ *pass through the points of* Q *for which* $t = 1$ *and* $t = p$. *Then*

$$\text{I} \equiv \int_1 t^{2r-1}Q(t)dt \geqq \int_1 {}_k t^{2r-1}ydt \equiv \text{II},$$

if $r \geqq 1$, $k \geqq 2$, $p = 2rk/(2r + 1)$.

It will be observed that, by subtracting a constant from each side of the inequality, the proposition can be reduced to the case where $Q(k) = 0$. This being supposed done, let $h = Q(1)/Q(p)$, and note that by definition $p > 1$, $h \geqq 1$. After determination of the values of a and b in accordance with the conditions imposed on y in the hypothesis, II becomes

$$\text{II} = Q(p)\int_1^k t^{2r-1}\frac{hp - 1 + (1 - h)t}{p - 1}dt. \tag{1}$$

Consider now the function represented by the line tangent to the curve Q at $t = p$, that is,

$$z = Q(p) + Q'(p)(t - p). \tag{2}$$

Since Q is concave upwards it never crosses this tangent, and so $Q(t) \geqq z(t)$, and

$$\text{I} \equiv \int_1^k t^{2r-1}Q(t)dt \geqq \int_1^k t^{2r-1}z(t)dt \equiv \text{III}. \tag{3}$$

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* BIOMETRIKA, vol. 12 (1919), p. 284.

In order to prove that I $\geqq$ II it is by (3) sufficient to show that III $\geqq$ II, and the lemma will now be established in this manner. First, fix arbitrarily all the parameters that occur in III, viz.: p, r, k, $Q(p)$, and $Q'(p)$, thus fixing the value of III. The value of II is also thus fixed, except so far as variation due to h is concerned. But, from (1),

$$\frac{d(\mathrm{II})}{dh} = \frac{Q(p)(1-k)}{(p-1)(2r+1)} \leqq 0,$$

and therefore, so far as variation with h is concerned, II is greatest when h is least, that is, when $h = 1$. But, then, $Q(1) = Q(p)$, and, since $Q(t)$ is monotonic, $Q(t) = Q(p)$ throughout the interval $1 \leqq t \leqq p$, and so $Q'(p) = 0$. Substituting this value for $Q'(p)$ in (2) and (3), and $h = 1$ in (1), we learn finally that, for this special value of h which makes II greatest, II = III. So, in general, II $\leqq$ III.

THEOREM: (*a*) *Let $f(x)$ be any frequency distribution, and suppose the origin and units to be so chosen that zero is its mean value and*

$$P(x) = \int_x^\infty f(x)dx.$$

(*b*) *Let $f(x)$ be a monotonic, decreasing function of $|x|$ when $|x| \geqq cs$, $c \geqq 0$.* This is a real restriction on $f(x)$, which varies in its severity according to the value of c. It will be discussed later. Its general effect is to except distributions of more than one mode, notably the "U" distributions.

(*c*) *Let $f(x)$ be symmetrical with respect to the centroid ordinate.* This may be assumed without loss of generality, for, if $g(x)$ is a skew distribution whose mean is at $x = 0$, it is clear that $f(x) = \frac{1}{2}[g(x) + g(-x)]$ has the same mean, even moments, and the same values of $P(\lambda s)$ as $g(x)$. *Then*

$$P(\lambda s) \leqq \frac{\beta_{2r-2}}{\left(\lambda\frac{2r+1}{2r}\right)^{2r}} \cdot \frac{1}{1+\phi} + \theta, \tag{4}$$

where

$$\theta = \frac{P(cs)}{1 + \dfrac{1}{\phi}}, \qquad \phi = \frac{\left(\dfrac{c}{\lambda} \cdot \dfrac{2r}{2r+1}\right)^{2r}}{(2r+1)\left(\dfrac{\lambda}{c} - 1\right)}, \tag{5}$$

quantities which can be tabulated and are usually very small.

Since $c \geqq 0$, we may write

$$\begin{aligned}\frac{\beta_{2r-2}}{c^{2r}} \geqq \int_{cs}^{\infty} \left(\frac{x}{cs}\right)^{2r} f(x)dx &= -\left[\left(\frac{x}{cs}\right)^{2r} \int_x^{\infty} f(t)dt\right]_{cs}^{\infty} \\ &\quad + \int_{cs}^{\infty} \frac{2rx^{2r-1}}{(cs)^{2r}}\, dx \int_x^{\infty} f(t)dt \\ \geqq P(cs) + \frac{2r}{cs}\int_{cs}^{kcs}\left(\frac{x}{cs}\right)^{2r-1} P(x)dx &= P(cs) + 2r\int_1^k t^{2r-1}Q(t)dt,\end{aligned}$$

where $x = tcs$, $Q(t) = P(tcs)$. Therefore, by the lemma, and by use of the relation $hQ(p) = Q(1)$,

$$\begin{aligned}\frac{\beta_{2r-2}}{c^{2r}} &\geqq P(cs) + \frac{2rQ(p)}{p-1}\int_1^k t^{2r-1}[(hp-1) + (1-h)t]dt \\ &= \frac{P(cs)}{1 + 2r - 2rk} \\ &\qquad + P(pcs)\left[\frac{(2r+1)(1-k^{2r}) + 2r(k^{2r+1}-1)}{1 + 2r - 2rk}\right].\end{aligned}$$

After the substitutions $k = p(2r+1)/2r$, $p = \lambda/c$, and some reductions, the result (4) follows from the above inequality. The assumption (*b*) will now be discussed.

CASE I: ($c = 0$). It was stated under assumption (*c*) that $f(x)$ might be assumed symmetrical, because a symmetrical function $f(x)$ having the same essential characteristics could be constructed from any skew function $g(x)$ that might arise in practice. It should be noticed, however, that if $g(x)$ is skew, the resulting $f(x)$ may be saddle shaped, with a geometrical minimum at its mean and two symmetrically situated maxima on either side. In such a case, the value of c would equal the distance from the mean to either maximum, s being the unit of measurement. In other words, if skewness be defined as (mean − mode)/s, c *may* be as large as the absolute value of the skewness. Now, skew functions are usually the ones that occur in practice, and so one could not usually assert, a priori, that $c = 0$ for a given distribution. This embarrassment might be avoided by choosing the origin at the mode of the skew function instead of at the mean, and by defining the moments and $P(\lambda s)$ with reference to such an origin, in which case, it could be shown, the above theorem would apply, and, for unimodal functions, c would equal zero.[*] But this is often impracticable, and certainly not customary. So it will be necessary to consider larger values of c. The following table for the case $c = 0$ is of value, however, since it is involved in the computation for the cases where $c > 0$. When $c = 0$, $\phi = \theta = 0$, and formula (4) is exactly Pearson's, except for the factor $[(2r + 1)/2r]^{2r}$ in the denominator. The values of this factor are as follows:

Formula	r	Factor	Formula	r	Factor
β_0	1	2.25	β_6	4	2.56
β_2	2	2.44	β_8	5	2.59
β_4	3	2.52	β_{10}	6	2.60

CASE II: ($c = 1$). If functions of more than one mode be excepted, a skewness greater than unity is very rare. Therefore, this may be regarded as a strong case, and Case III below, where $c = 2$, as an extreme case. To compute formula (4) we may use the table under Case I for the first fraction, and the tables below for ϕ and θ. The values given for θ are really the extreme values which θ has when $P(cs = s) = 1$. It is unnecessary to estimate them more closely. In this case, the theorem will be found of value only when $\lambda > 1$, but this is not a serious misfortune, since in practice it is most needed for about the range $2 < \lambda < 4$.

r	θ			ϕ		
	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$
1	.015 6	.008 13	.004 28	.015 8	.008 23	.004 84
2	.001 40	.000 51	.000 22			
3	.000 15	.000 039	.000 012	ϕ equals θ to the number of places indicated		
4	$1.9 \cdot 10^{-5}$	$3.8 \cdot 10^{-6}$	$7.7 \cdot 10^{-7}$			

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[*] While this paper was in press a paper by B. Meidel appeared, COMPTES RENDUS, vol. 175, p. 806 (Nov. 6, 1922), giving the result of this paper for the case $c = 0$, and using the mode as origin.

CASE III: ($c = 2$). The computation in this case will again be performed by use of the table under Case I, and of the subjoined table for ϕ and for $\theta/P(cs = 2s)$. Then Pearson's inequality should be used to estimate $P(2s)$. In this case the theorem will be found of value only when $\lambda > 2$.

r	$\theta/P(2s)$			ϕ		
	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$
1	.275	.116	.060 6	.379	.132	.065
2	.118	.031 4	.011 5	.134	.032	.012
3	.056 1	.009 88	.002 63	.059	.010	.003
4	.028 3	.003 38	.006 58	.029	.003	.001

3. *Example.* In BIOMETRIKA * the frequency distributions of a number of coefficients of correlation are actually computed. One may apply the above inequalities to these distributions (using the β's as if they only, not the entire frequency distributions, had been calculated), and then compare the results obtained with the true values of $P(\lambda s)$ to be found by partially summing the frequencies there tabulated. Take $\lambda = 3$ in the distribution for which $\rho = 0.8$, $n = 100$, on page 403. If the distribution be regarded as a set of rectangles, the true value of $P(3s)$ is about 0.009. The β_2 formula shows that $P(3s) \leqq 0.018$, and the β_4 formula that $P(3s) \leqq 0.012$. Case II has been used on the theory that the practical statistician would be confident that $c \leqq 1$, but the use of Case III would increase the better of the two results given to only 0.014. Here, although β_2 departs from its Gaussian value by only 0.42, it is not safe to use the Gaussian table, which would make $P(3s) = 0.0027$. The reason is that the higher β's depart much more widely from their Gaussian values. They are, roughly, $\beta_2 = 3.42$, $\beta_4 = 22.7$, $\beta_6 = 290$, $\beta_8 = 5000$, $\beta_{10} = 100\,000$; instead of 3, 15, 105, 945, and 10 395, respectively. It is a common, but unwarranted and sometimes very faulty practice, to use the Gaussian where β_2 only is near its Gaussian value.

WESLEYAN UNIVERSITY

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* Vol. 11 (1915–17), pp. 379–404.

TWO THEOREMS ON MULTIPLE INTEGRALS*

BY PHILIP FRANKLIN

1. *Introduction.* The multiple integrals in question here were discussed by Poincaré (ACTA MATHEMATICA, vol. 9, p. 321) in a paper in which he defined them and derived their integrability conditions. In this note we give two theorems, which simplify the derivation of these conditions, and enable us to express the integral of an integrable function over an open region as an integral of one order lower over the boundary of this region. The second theorem was suggested by a special case proved by Professor Eisenhart.

2. *First Theorem.* Our first theorem is

$$\int n \int \sum A_{i_1\ldots i_n} dx_{i_1}\cdots dx_{i_n} \qquad (1)$$

$$\frac{1}{m-n}\int n+1 \int \sum \frac{\partial A_{i_1\ldots i_n}}{\partial x_{i_{n+1}}} dx_{i_{n+1}} dx_{i_1}\cdots dx_{i_n}.$$

where the n (dimensional) region of integration for the left member is the boundary of the $n+1$ region for the right member, and the summations apply to all n or $n+1$ permutations of the m subscripts of our fundamental space. The theorem is proved by adding equations of the form

$$\int n \int A_{1\ldots n} dx_1\cdots dx_n \qquad (2)$$

$$= \int n+1 \int \frac{\partial A_{1\ldots n}}{\partial x_{n+1}} dx_{n+1} dx_1 \cdots dx_n,$$

which may be established when the A's are functions of m variables by methods similar to those ordinarily used to prove (2) in n-space. The equation (1) shows that the set of equations

$$\sum_j (-1)^{j-1} \frac{\partial A_{i_1\ldots i_{j-1} i_{j+1}\ldots i_{n+1}}}{\partial x_{i_j}} = 0 \qquad (3)$$

is a necessary and sufficient condition for the vanishing of the left member of (1) for all closed regions of integration, i.e., that (3) is the condition of integrability for the form

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* Presented to the Society, September 7, 1922.

$$\sum A_{i_1\ldots i_n} dx_{i_1}\cdots dx_{i_n}. \tag{4}$$

3. *Second Theorem.* It follows from (1) also that the integral

$$\int n+1 \int \sum B_{i_1\ldots i_{n+1}} dx_{i_1}\cdots dx_{i_{n+1}}, \tag{5}$$

taken over any open region may be expressed as an integral

$$(m-n)(n+1)\int n \int \sum A_{i_1\ldots i_n} dx_{i_1}\cdots dx_{i_n}, \tag{6}$$

taken over its boundary, provided A's can be found such that

$$B_{i_1\ldots i_{n+1}} = \sum_j (-1)^{j-1} \frac{\partial A_{i_1\ldots i_{j-1} i_{j+1}\ldots i_{n+1}}}{\partial x_j}. \tag{7}$$

We shall show that *a necessary and sufficient condition for the existence of such a set of A's is that* (5) *be integrable,* i.e., that

$$\sum (-1)^{j-1} \frac{\partial B_{i_1\ldots i_{j-1} i_{j+1}\ldots i_{n+2}}}{\partial x_j} = 0. \tag{8}$$

This theorem is known for the case $n = 0$. We shall outline an induction by which the general case may be reduced to this one. We first consider the case where there are $n + 2$ i's, and hence $n + 2$ equations (7) and one condition (8). If we write

$$B_{1\ldots p-1,\, p+1\ldots n+2} = B'_{1\ldots p-1,\, p+1\ldots n+1} + (-1)^n \frac{\partial A_{1\ldots p-1,\, p+1\ldots n+1}}{\partial x_{n+2}}, \tag{9}$$

where $p \neq n + 2$, and where the B''s are defined by the corresponding equation (7), and if we determine the A's appearing explicitly in (9) so as to satisfy the remaining equation of the set (7),

$$B_{1\ldots n+1} = \sum_j (-1)^{j-1} \frac{\partial A_{1\ldots j-1,\, j+1\ldots n+1}}{\partial x_j}, \tag{10}$$

but leave their values otherwise arbitrary, we find that by the use of (9) and (10) the condition (8) reduces to the form

$$\sum_j (-1)^{j-1} \frac{\partial B'_{1\ldots j-1,\, j+1\ldots n+1}}{\partial x_j} = 0. \tag{11}$$

Since (11) is of the same form as (8), but in one less index, and since the expressions for the B's in terms of A's are similar to (7), we have reduced the problem to one of lower order.

To complete our proof, we reduce the case for $m+1$ subscripts to that for m. We first obtain values for the A's not involving one subscript, say k, from the equations and conditions not involving k, by the case assumed as the basis of the induction. Then we make the substitutions:

$$B_{i_1\ldots i_n k} = B'_{i_1\ldots i_n} + (-1)^n \frac{\partial A_{i_1\ldots i_n}}{\partial x_k}, \tag{12}$$

which effects the desired reduction.

This proves the sufficiency of the conditions; that they are necessary follows by direct substitution. A more complete discussion of the properties of multiple integrals is given in an expository article that will appear in the ANNALS OF MATHEMATICS.

HARVARD UNIVERSITY

KIRKMAN PARADES*

BY F. N. COLE

On examining the complete list of non-equivalent triad systems in 15 letters published in the MEMOIRS OF THE NATIONAL ACADEMY OF SCIENCES (vol. 14, No. 2, pp. 77–80), it appears that only four of these are Kirkman systems, if this name be applied to those cases where the 35 triads divide into seven sets (or columns) of five with each column containing all the 15 letters. Such a seven-column arrangement might be called a Kirkman parade. And it turns out that three of the Kirkman systems give each two non-equivalent parades, while the fourth system gives only one parade.

Kirkman proposed his problem in the LADY'S AND GENTLEMAN'S DIARY for 1850. The seven solutions were correctly given by Woolhouse in the same Diary for 1862 and 1863. In 1881 Carpmael published a list of eleven solutions in the PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY (vol. 12, pp. 148–156). But his sixth and seventh items duplicate the third and fourth, and the fifth and eleventh duplicate the ninth and tenth.

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* Presented to the Society, September 7, 1922.

In forming the columns of a parade, any two of them can always be reduced to one of the types

$$(\alpha)\quad \begin{array}{ccc}1&2&3\\4&5&6\\7&8&9\\10&11&12\\13&14&15\end{array}\qquad \begin{array}{ccc}1&4&7\\2&5&8\\3&10&13\\6&11&14\\9&12&15\end{array}\qquad\qquad (\beta)\quad \begin{array}{ccc}1&2&3\\4&5&6\\7&8&9\\10&11&12\\13&14&15\end{array}\qquad \begin{array}{ccc}1&4&7\\2&5&10\\3&8&13\\6&11&14\\9&12&15\end{array}$$

We may say that two columns are *laced* in the mode (α) or the mode (β), and the seven parades are readily distinguished by this lacing: The parades follow in tabular form, together with the substitution group of each system and parade and the scheme of interlacing of the columns.

I–II

1	2	3	1	4	7	1	5	15	1	9	13	1	6	10	1	8	11	1	12	14
4	5	6	2	5	8	2	9	10	2	4	12	2	11	15	2	7	14	2	6	13
7	8	9	3	10	13	3	4	14	3	5	11	3	7	12	3	6	9	3	8	15
10	11	12	6	11	14	6	8	12	6	7	15	4	8	13	4	10	15	4	9	11
13	14	15	9	12	15	7	11	13	8	10	14	5	9	14	5	12	13	5	7	10
												1	6	10	1	8	11	1	12	14
												2	7	14	2	6	13	2	11	15
	“			“			“			“		3	8	15	3	7	12	3	6	9
												4	9	11	4	10	15	4	8	13
												5	12	13	5	9	14	5	7	10

These two parades are included in a single triad system, which contains 120 conjugates of each of them. The group of the system is of order $8!/2$. The group of I is of order 168 and is generated by $(1\ 11\ 7\ 5\ 9\ 12\ 2)(3\ 8\ 13\ 10\ 14\ 15\ 4)$ and $(1\ 13\ 12\ 11\ 15\ 3\ 4)(2\ 8\ 14\ 7\ 9\ 5\ 10)$; it is transitive in all the letters but 6. The group of II is also of order 168, but it is transitive in seven and in eight letters, being generated by $(1\ 2\ 9\ 13\ 6\ 3\ 10)(4\ 14\ 8\ 11\ 15\ 12\ 5)$ and $(1\ 10\ 13\ 9\ 2\ 6\ 3)(4\ 8\ 7\ 14\ 11\ 5\ 15)$.

In both I and II the columns have only the lacing (α).

III–IV

1	2	3	1	4	7	1	5	15	1	9	10	1	6	13	1	8	11	1	12	14
4	5	6	2	5	8	2	9	13	2	4	12	2	7	14	2	6	10	2	11	15
7	8	9	3	10	13	3	4	11	3	5	14	3	8	12	3	7	15	3	6	9
10	11	12	6	11	14	6	7	12	6	8	15	4	10	15	4	9	14	4	8	13
13	14	15	9	12	15	8	10	14	7	11	13	5	9	11	5	12	13	5	7	10
												1	6	13	1	12	14	1	8	11
												2	11	15	2	6	10	2	7	14
	“			“			“			“		3	8	12	3	7	15	3	6	9
												4	9	14	4	8	13	4	10	15
												5	7	10	5	9	11	5	12	13

These are included in a single triad system, which contains 12 conjugates of each of them. The group of the system is of order 288 The group of III is of order 24 and is generated by (1 8 5)(2 15 11)(3 6 9)(4 10 12)(7 14 13) and (1 10 2 13)(5 15 4 12)(7 11 8 14)(6 9). The group of IV is also of order 24 and is generated by (1 10 13)(3 6 9)(4 5 11)(8 15 12) and (1 12 2 15)(4 11 5 14)(7 10 8 13)(3 9). The last column in each case is in relation (α) to all the other columns; each of the latter has two (α) lacings and four (β) lacings.

V–VI

1	2	3	1	4	7	1	5	13	1	6	8	1	9	10	1	11	15	1	12	14
4	5	6	2	5	10	2	4	12	2	11	13	2	7	14	2	6	9	2	8	15
7	8	9	3	8	13	3	9	11	3	7	12	3	5	15	3	4	14	3	6	10
10	11	12	6	11	14	6	7	15	4	10	15	4	8	11	5	8	12	4	9	13
13	14	15	9	12	15	8	10	14	5	9	14	6	12	13	7	10	13	5	7	11
												1	9	10	1	11	15	1	12	14
												2	8	15	2	7	14	2	6	9
	“			“			“			“		3	4	14	3	6	10	3	5	15
												5	7	11	4	9	13	4	8	11
												6	12	13	5	8	12	7	10	13

These again are both contained in one triad system. They have both the same group as the system, viz., the tetrahedral group generated by (1 12)(2 7)(5 15)(8 11)(6 10)(9 13) and (1 2 3)(4 12 7)(6 10 9)(5 11 8). The last three columns of each have the lacing (α); the other lacings are all of type (β).

VII

1	2	3	1	4	7	1	5	9	1	6	15	1	8	11	1	10	13	1	12	14
4	5	6	2	5	10	2	4	14	2	9	11	2	6	13	2	7	12	2	8	15
7	8	9	3	8	13	3	10	15	3	7	14	3	4	12	3	6	9	3	5	11
10	11	12	6	11	14	6	8	12	4	8	10	5	7	15	4	11	15	6	7	10
13	14	15	9	12	15	7	11	13	5	12	13	9	10	14	5	8	14	4	9	13

This has a group of order 21, generated by (1 2 12 3 7 4 14) (5 8 11 9 15 10 6) and (2 7 12)(3 4 14)(5 6 11)(8 9 15), which is also the group of the triad system. The lacings of the columns are all of type (β).

COLUMBIA UNIVERSITY

IMPOSSIBILITY OF RESTORING UNIQUE FACTORIZATION IN A HYPERCOMPLEX ARITHMETIC

BY L. E. DICKSON

1. *Introduction.* Most numbers $a + be$, where a and b are integers and $e^2 = 0$, admit of several factorizations into indecomposable numbers. It is proved in § 3 that we cannot restore unique factorization by defining hypercomplex ideals analogous to algebraic ideals, nor (§ 4) by the introduction of any sort of ideals obeying the laws of arithmetic. L. G. du Pasquier* has made statements, omitting proofs, concerning the failure of unique factorization after introducing ideals, apparently meaning those analogous to algebraic ideals.

2. *Hypercomplex Integers.* Consider the hypercomplex numbers $x = a + be$ with rational coordinates a, b, where $e^2 = 0$. Thus $(x - a)^2 = 0$. This quadratic equation has integral coefficients if and only if a is integral. As our integral hypercomplex numbers we shall take those of an infinite system of numbers $a + be$, where a is integral and b rational, such that the system has a basis† 1, ce, i.e., is composed of their linear combinations with integral coefficients. Since we may take ce as a new unit e whose square is zero, we may assume that $a + be$ is integral if and only if a and b are both integers.

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* Vierteljahrsschrift, Zürich, vol. 54 (1909), pp. 116–148.
L'Enseignement, vol. 17 (1915), pp. 340–3; vol. 18 (1916), pp. 201–260.
Nouvelles Annales, (4), vol. 18 (1918), pp. 448–461.
Comptes Rendus du Congrès International (Strasbourg), 1921.

† We obtain uninteresting results if we omit the assumption of a basis and call $a + be$ integral if a is integral and b rational. It is a unit if $a = \pm 1$. If $r \neq 0$, $\rho = r + se$ is "associated" with its product r by the unit $1 - es/r$. Hence the classes of associated numbers whose real coordinates are not zero are in (1, 1) correspondence with the real integers and obey the laws of divisibility of integers. But se is associated only with $\pm se$. Now te is divisible by every $r + se$, $r \neq 0$, the quotient being et/r.

The units (integral divisors of 1) are $\pm 1 + be$, where b is integral. A simple example shows that the laws of divisibility fail. Let p be any odd prime. Then

$$p \cdot p = (p + ke)(p - ke), \tag{1}$$

and $p + te$ is indecomposable and is associated with $p + le$ if and only if $t \equiv l \pmod{p}$. The product of $p - ke$ by the unit $1 + e$ is $p + (p - k)e$. Hence if we give to k the values $0, 1, \cdots, \frac{1}{2}(p - 1)$ in (1), we obtain the $\frac{1}{2}(p + 1)$ essentially different ways of factoring p^2 into indecomposables. As a generalization of (1),

$$(p + ke)(p + le) = (p + xe)(p + ye), \qquad x + y = k + l.$$

3. *Hypercomplex Ideals.* As in the theory of algebraic numbers, define an ideal to be an infinite set of our hypercomplex integers which is closed under addition and subtraction and is such that the product of any number of the set by any hypercomplex integer is equal to a number of the set. Since $(r + se)e = re$, every ideal contains a number te, where $t \neq 0$. Let m be the minimum positive integer such that me is in the ideal. The products xme of me by all hypercomplex integers $x + ye$ constitute a principal ideal, denoted by $[me]$. Consider an ideal I which contains xme, where x ranges over all integers, and further numbers $r_i + s_i e$, where each $r_i \neq 0$. The positive greatest common divisor r of the r_i is a linear combination of them with integral coefficients. The same linear combination of the $r_i + s_i e$ is a number $r + se$ of I. Write q_i for the integer r_i/r. Then I contains

$$r_i + s_i e - q_i(r + se) = k_i e, \qquad k_i \equiv s_i - q_i s.$$

Hence $k_i = p_i m$, where p_i is an integer. Thus

$$r_i + s_i e = p_i(me) + q_i(r + se),$$

so that I is composed of linear combinations of me and $r + se$ with integral coefficients (i.e., has a basis). Finally, the product of every number of I by every hypercomplex integer must belong to I, which will be true if the products by e belong to I, i.e., if re is in I. Thus $r = qm$, where q is an integer. Hence

$$I = [me,\ qm + se], \qquad q > 0,\ m > 0,\ 0 \leqq s < m, \tag{2}$$

where the bracket signifies all the linear combinations of the two enclosed numbers with integral coefficients.

The product of two ideals is defined to be the totality of linear combinations with integral coefficients of the various products of numbers of the first ideal by numbers of the second. Hence

$$[te]I = [tqme] = [te]P, \qquad P \equiv [qme,\ qm].$$

The ideal P is distinct from I unless $q = 1$, $s = 0$. Hence the laws of arithmetic do not hold for our ideals.*

The preceding special difficulty may be obviated by excluding one-based ideals $[te]$. Hence we shall supplement our definition of an ideal by making the assumption that it contains numbers which are not divisors of zero, i.e., multiples of e. Now every ideal is of the form (2).

If $q = 1$, (2) is composed of the products of $m + se$ by all hypercomplex integers and hence is called a principal ideal $\{m + se\}$. If $q > 1$, (2) is not a principal ideal. We shall call q the *mass* of the ideal (2). Hence an ideal is a principal ideal if and only if its mass is unity.

The mass of a product of two ideals is the least common multiple of their masses. For, the product of (2) by

$$J = [ne,\ pn + te]$$

is

$$IJ = [p \cdot mne,\ q \cdot mne,\ pqmn + ke], \qquad k = qmt + pns.$$

Let G be the greatest common divisor of $p = PG$ and $q = QG$, whence P and Q are relatively prime. Then G is a linear combination of p and q. The same linear combination of the first two entries in IJ is $Gmne$. Hence

$$IJ = [Gmne,\ PQGGmn + ke],$$

whose mass is PQG, i.e., the least common multiple of p and q.

It follows at once that the mass of the product of any number of ideals is the least common multiple of their masses. The latter is unity if and only if all the masses are unity. Hence a product is a principal ideal if and only if each factor is a principal ideal. In other words, every factor of a principal ideal is itself a principal ideal.

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* More simply since $[e][e]$ is not an ideal.

Consequently our ideals fail to explain our difficulty (1), or the equivalent in principal ideals:

$$(1') \qquad \{p\}\cdot\{p\} = \{p + ke\}\{p - ke\}.$$

Since we saw that $p + te$ is an indecomposable number, we conclude that $\{p + te\}$ is not a product of principal ideals and not a product of any ideals. The relation (1′) between distinct indecomposable ideals shows that our ideals do not obey the laws of arithmetic and do not explain our difficulty (1).

4. *Impossibility of the Restoration of Unique Factorization.* In § 3 we saw the futility of the introduction of hypercomplex ideals defined essentially as in the theory of algebraic numbers. We shall now prove that it is impossible to restore unique factorization by the introduction of ideals of any kind such that a number and its products by the units all correspond to a unique ideal (provided the number be not a divisor of zero) and such that the product of two numbers corresponds to the product of the corresponding ideals.

The numbers $a = 3$, $b = 3 + e$, $c = 3 + 2e$ are indecomposable and no two are associated. We have

$$(3) \qquad ac = b^2, \qquad bc = a^2(1 + e), \qquad ab = c^2(1 - e).$$

Let α, β, γ be the distinct ideals $\neq 1$ which correspond uniquely to a, b, c, respectively. Since $1 + e$ and $1 - e$ are units, we have

$$(4) \qquad \alpha\gamma = \beta^2, \qquad \beta\gamma = \alpha^2, \qquad \alpha\beta = \gamma^2.$$

A prime ideal divisor $\delta \neq 1$ of α must divide β and γ. Write

$$\alpha = \alpha_1\delta,\ \beta = \beta_1\delta,\ \gamma = \gamma_1\delta.$$

Then

$$(5) \qquad \alpha_1\gamma_1 = \beta_1^2, \qquad \beta_1\gamma_1 = \alpha_1^2, \qquad \alpha_1\beta_1 = \gamma_1^2.$$

No one of α_1, β_1, γ_1 is unity. For, if $\alpha_1 = 1$, then $\beta_1\gamma_1 = 1$, whence $\beta_1 = \gamma_1 = 1$, whereas α, β, γ are distinct. Since the relations (5) are entirely similar to relations (4), it is impossible to restore, in a finite number of steps, unique factorization in (3) by the introduction of ideals.

5. *Factorization.* There is only a finite number of ways of factoring into indecomposables and each way involves only a finite number of factors. First, to obtain all pairs of factors of te, express t as a product of two integers x, w in all possible ways; the factors are $x + ye$, we, where $0 \leqq y < x$, if we retain only one of associated numbers. To factor $a + be$, where $a > 0$, express a in all possible ways as a product xz of two integers > 1. For each such pair x, z, and for $0 \leqq y < x$, $a + be$ has the factors $x + ye$ and $z + we$, where w is determined uniquely by $xw + yz = b$, the case of a fractional w being excluded.

The only pairs of factors of p^k $(k \geqq 2)$, no one a unit, are

$$p^r + ye, \qquad p^{k-2r}(p^r - ye),$$

where r ranges over the positive integers $\leqq \frac{1}{2}k$, and $0 \leqq y < p^r$. The only pairs of factors of $p^k + se$ are

$$p^r + ye, \qquad p^{k-r} + (\sigma - p^{k-2r}y)e,$$

where $r \leqq \frac{1}{2}k$ and p^r divides $s = \sigma p^r$.

If p and q are distinct primes and $k \geqq l$, the only pairs of factors not units of $p^k q^l$ are J, $p^{k-2r}q^{l-2s}K$, where

$$J = p^r q^s + ye, \qquad K = p^r q^s - ye,$$

$0 \leqq r \leqq \frac{1}{2}k$, $0 \leqq s \leqq \frac{1}{2}l$, $0 \leqq y < p^r q^s$; and $p^{k-2r}J$, $q^{l-2s}K$ $(r < \frac{1}{2}k)$. In particular pq has only the factors p, q.

Regarding factors of J, note that $pq + se$ has the unique factors, apart from unit factors, $p + ye$ and $q + we$, where $0 \leqq y < p$, $qy \equiv s \pmod{p}$, and w is determined by $pw + yq = s$. Next, $p^2 q + se$ has the factors $p^2 + ye$ and $q + we$, where w is uniquely determined by $0 \leqq w < q$, $p^2 w \equiv s \pmod{q}$, and y is then determined by $p^2 w + qy = s$. It has no further pairs of factors if s is prime to p. But if $s = Sp$, the only additional pairs are the p pairs $p + ye$, $pq + we$, where $0 \leqq y < p$, $w = S - qy$.

THE UNIVERSITY OF CHICAGO

A REVISION OF THE BERNOULLIAN AND EULERIAN FUNCTIONS*

BY E. T. BELL

1. *Introduction.* To secure simplicity and uniformity in the derivation of relations between the Bernoullian and Eulerian *functions* (not to be confused with the *numbers* of the same name) occurring most frequently in applications we take as fundamental a set of four functions instead of the usual pair, and from these by easy substitutions obtain the values of the functions defined by other writers.†

Following Lucas ‡ we use the even suffix notation for the numbers B, G, E, R of Bernoulli, Genocchi, Euler and Lucas; with the exceptions $B_1 = -\frac{1}{2}$, $G_1 = 1$, the numbers of odd rank vanish, and $B_0 = 1$, $G_0 = 0$, $E_0 = 1$, $R_0 = \frac{1}{2}$. Unless otherwise stated n is an arbitrary integer $\geqq 0$. The "representative" or umbral calculus of Blissard § is used throughout, so that the nth power a^n of the umbra a represents the ordinary a_n. The letters x, y, z, u, v denote ordinary algebraic quantities, or ordinaries; a, b, c, B, E, G, R, φ, ψ, λ, μ are umbræ. The umbral sin ax, cos ax are defined by the series, assumed absolutely convergent for some $|x| > 0$,

$$\sin ax = \sum_0^\infty (-1)^n a^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_0^\infty (-1)^n a_{2n+1} \frac{x^{2n+1}}{(2n+1)!},$$

$$\cos ax = \sum_0^\infty (-1)^n a^{2n} \frac{x^{2n}}{(2n)!} = \sum_0^\infty (-1)^n a_{2n} \frac{x^{2n}}{(2n)!};$$

the umbral multinomial theorem gives the expansion of $(ax + by + \cdots + cz)^n$ in either of the identical forms

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[*] Presented to the Society, April 8, 1922.

[†] Accounts of the history and notations of these functions are contained in papers by Worpitzky, CRELLE'S JOURNAL, vol. 94 (1883), pp. 203–232, and Glaisher, QUARTERLY JOURNAL, vol. 29 (1898), pp. 1–169; vol. 42 (1911), pp. 86–157.

[‡] *Théorie des Nombres*, Chap. XIV.

[§] QUARTERLY JOURNAL, vols. 6–9 (1863–1867); cf. also Lucas, loc. cit., Chap. XIII.

$$\sum \frac{n!}{\alpha!\beta!\cdots\gamma!} x^\alpha y^\beta \cdots z^\gamma a^\alpha b^\beta \cdots c^\gamma,$$

$$\sum \frac{n!}{\alpha!\beta!\cdots\gamma!} x^\alpha y^\beta \cdots z^\gamma a_\alpha b_\beta \cdots c_\gamma,$$

the $\sum$ extending to all sets of values of $\alpha, \beta, \cdots, \gamma \geqq 0$ whose sum is n. When precisely r of the umbræ $a, b, \cdots, c$ are the same they are replaced by r distinct umbræ until after the degradation of exponents. Thus $(a - a)^2$ is obtained from $(a - b)^2$ or $a^2b^0 - 2a^1b^1 + a^0b^2$, and its value is $2(a_0a_2 - a_1^2)$. Throughout the paper $f(x)$ is an arbitrary analytic function of x; its derivatives with respect to x are denoted by accents, $f'(x), f''(x), \cdots$; with respect to any other variable, say u, of which it is a function, by $f_u, f_{uu}, \cdots$. As a special case of importance in applications the power series representation $k_0 + k_1x + k_2x^2 + \cdots$ of $f(x)$ may terminate; that is, for m a finite constant, $k_j = 0, j > m$.

2. *Umbral Identities.* The functions defined by

$$\varphi_n(a, b) \equiv (a + b)^n + (a - b)^n,$$
$$\psi_n(a, b) \equiv (a + b)^n - (a - b)^n,$$

and the identities connecting them, are fundamental for the whole subject.* Each is said to be of rank n. Noting that they may be written umbrally $\varphi^n(a, b)$, $\psi^n(a, b)$, we have

$$(1) \quad \begin{aligned} 2 \cos ax \cos bx &= \cos \varphi(a, b)x, \\ 2 \sin ax \sin bx &= -\cos \psi(a, b)x; \end{aligned}$$

$$(2) \quad \begin{aligned} 2 \sin ax \cos bx &= \sin \varphi(a, b)\, x, \\ 2 \cos ax \sin bx &= \sin \psi(a, b)x. \end{aligned}$$

From the definitions we write down as special cases the values of $\varphi_n(xa, yb)$, $\psi_n(xa, yb)$, $\varphi_n(x, yb)$, etc. For, if $a \equiv yb$, then

$$a_n = a^n = y^nb^n = y^nb_n,$$

since y is an ordinary, b an umbra; while if $a \equiv x$,

$$a_n = a^n = x^n.$$

The suffix u denoting as above the u-derivative, we have

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* This is true whatever definition of the Bernoullian and Eulerian functions be taken. Thus the existing treatments could be much simplified by the use of φ, ψ. It appears best, however, to recast the entire theory.

$$(3) \qquad \lambda_u{}^n(u, a) = n\lambda^{n-1}(u, a), \qquad \lambda = \varphi, \psi;$$

and it is easily seen that

$$(4) \quad \varphi^{n+1}(u, a) = (n+1)\int_0^u \varphi^n(u, a)du + [1-(-1)^n]a_{n+1}.$$

Let ϵ denote either $+1$ or -1. Then

$$\begin{aligned}(x + \epsilon a + \epsilon b)^n \pm (x + \epsilon a - \epsilon b)^n &= (x + \epsilon\lambda(a, b))^n \\ &= \sum_{r=0}^{n} \binom{n}{r} \epsilon^r \lambda^r(a, b)x^{n-r},\end{aligned}$$

where $\lambda = \varphi$ or ψ according as the upper or lower sign on the left is taken. Multiply this throughout by $k_0, k_1, k_2, \cdots$ and add all the results:

$$(5) \quad f(x + \epsilon a + \epsilon b) \pm f(x + \epsilon a - \epsilon b) = f(x + \epsilon\lambda(a, b)),$$

the ambiguous sign being determined as above. In the same way,

$$\begin{aligned}(6) \quad f(x + a + b) + \epsilon_1 f(x + a - b) + \epsilon_2 f(x - a - b)& \\ + \epsilon_1\epsilon_2 f(x - a + b) &= f(\mu(x, \lambda)),\end{aligned}$$

where $\lambda = \varphi(a, b)$ or $\psi(a, b)$ according as $\epsilon_1 = +1$ or -1, and $\mu = \varphi$ or ψ according as $\epsilon_2 = +1$ or -1. Many more relations of a similar kind can be written down in the like manner. The process by which (5), (6) are obtained is more easily applied in many cases than the final formulas. For example, suppose we have formed the relation $(a + b)^{2n} = c^{2n}$. Then since n is an arbitrary integer $\geqq 0$ we have identically

$$(x + a + b)^n + (x - a - b)^n = (x + c)^n + (x - c)^n,$$

and from this, proceeding as before,

$$f(x + a + b) + f(x - a - b) = f(x + c) + f(x - c).$$

All of the above formulas can be easily generalized to the case of any number n of umbræ by extending (1), (2) to n factors on the left. In doing this it is advantageous to take as fundamental the function generated by the product of n_1 umbral sines and n_2 umbral cosines, $n_1 + n_2 = n$. The cases $(n_1, n_2) = (1, 1), (2, 0), (0, 2)$ are sufficient for the present sketch.

From Lucas,* we transcribe the relations

$$(7)\qquad x \operatorname{ctn} x = \cos 2Bx, \qquad 2x \tan x = \cos 2Gx,$$

$$(8)\qquad \sec x = \cos Ex, \qquad x \csc x = 2 \cos Rx.$$

Although the lack of symmetry in these destroys many uniformities later we shall retain the definitions of B_n, G_n, E_n, R_n implied in them, as Lucas' notation is well established.

3. *The Fundamental Functions.* The Bernoullian functions $\beta^n(u) \equiv \beta_n(u)$, $\gamma^n(u) \equiv \gamma_n(u)$ of rank n and argument u are now defined by

$$(9)\qquad \beta^n(u) = \varphi^n(u, 2B) = 2\sum_{r=0} 2^{2r}\binom{n}{2r} B_{2r}\, u^{n-2r},$$

$$(10)\qquad \gamma^n(u) = \varphi^n(u, 2G);$$

and the Eulerian functions similarly by

$$(11)\qquad \eta^n(u) = \varphi^n(u, E) = 2\sum_{r=0}\binom{n}{2r} E_{2r}\, u^{n-2r},$$

$$(12)\qquad \rho^n(u) = \varphi^n(u, R).$$

When there can be no confusion the argument u will be omitted but understood, and we shall write $\lambda_n(u) \equiv \lambda_n$ $(\lambda = \beta, \gamma, \eta, \rho)$. The derivatives are obtained from (3),

$$(13)\qquad \lambda_u^n = n\lambda^{n-1}, \qquad (\lambda = \beta, \gamma, \eta, \rho);$$

and from (9)–(12), noting the zero values of B, G, E, R to write down those of β, γ, η, ρ, we have for $(\lambda, L) \equiv (\beta, B)$, (γ, G) and $(\mu, M) \equiv (\eta, E)$, (ρ, R),

$$(14)\qquad \begin{aligned} \lambda_{2n+1} &= (2n+1)\int \lambda_{2n}du, \\ \lambda_{2n+2} &= (2n+2)\int \lambda_{2n+1}du + 2^{2n+3} L_{2n+2}; \end{aligned}$$

$$(15)\qquad \begin{aligned} \mu_{2n+1} &= (2n+1)\int \mu_{2n}du, \\ \mu_{2n+2} &= (2n+2)\int \mu_{2n+1}du + 2M_{2n+2}. \end{aligned}$$

With the initial values

$$(16)\qquad \beta_0 = 2, \qquad \gamma_0 = 0, \qquad \eta_0 = 2, \qquad \rho_0 = 1,$$

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* Loc. cit., p. 261.

these enable us to calculate the successive λ_n rapidly when the B, G, E, R numbers are given by either tables or recurrences. From (9)–(12), (7)–(8), (1), (2) we write down the generating identities,*

$$\sin \beta x = 2x \operatorname{ctn} x \sin ux, \qquad \cos \beta x = 2x \operatorname{ctn} x \cos ux; \tag{17}$$

$$\sin \gamma x = 4x \tan x \sin ux, \qquad \cos \gamma x = 4x \tan x \cos ux; \tag{18}$$

$$\sin \eta x = 2 \sec x \sin ux, \qquad \cos \eta x = 2 \sec x \cos ux; \tag{19}$$

$$\sin \rho x = x \csc x \sin ux, \qquad \cos \rho x = x \csc x \cos ux. \tag{20}$$

In conjunction with (1), (2), (7), (8) these contain the entire theory of the relations between β, γ, η, ρ. The relations are found uniformly from (17)–(20) by obvious trigonometric manipulations. There is space here only for a few examples to illustrate the processes.

4. *Relations between the Functions.* Multiply the identities (17) throughout by $4x \tan x$, use the second of (7) to reduce the left-hand members of the results to the form of the corresponding member of (2), and apply (2):

$$\sin \varphi(\beta, 2G)x = 8x^2 \sin ux, \qquad \cos \varphi(\beta, 2G)x = 8x^2 \cos ux.$$

Equate coefficients of like powers of x,

$$\varphi^n(\beta, 2G) = -8 \frac{d^2u^n}{du^2};$$

generalize this as in obtaining (5), noticing that $f''(x+u) = f_{uu}(x+u)$:

$$f(x+\beta+2G) + f(x+\beta-2G) = -8f''(x+u). \tag{21}$$

Similarly from (18),

$$f(x+\gamma+2B) + f(x+\gamma-2B) = -8f''(x+u). \tag{22}$$

If in this $f(x) \equiv x^n$, and then $x = 0$, the form from which it was inferred is recovered. In the same way from (19), (20) after multiplication throughout by $2 \cos x$, $2 \sin x$ respectively,

$$f(x+\eta+1) + f(x+\eta-1) = 4f(x+u), \tag{23}$$

$$f(x+\rho+1) - f(x+\rho-1) = 2f'(x+u); \tag{24}$$

and from (17), (18) for the multipliers $\sin x$, $\cos x$ we find first

$$\psi^n(\beta, 1) = 2\varphi_u{}^n(u, 1), \qquad \varphi^n(\gamma, 1) = -4\psi_u{}^n(u, 1),$$

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* Note the unnecessary lack of symmetry caused by the unfortunate definitions (7), (8).

whence by generalization to functions f,

$$(25)\quad f(x+\beta+1)-f(x+\beta-1)=2f'(x+u+1)+2f'(x+u-1),$$

$$(26)\quad f(x+\gamma+1)+f(x+\gamma-1)=4f'(x+u-1)-4f'(x+u+1).$$

Again, dividing each of the identities in each of (17)–(20) by the other we get, for $\lambda = \beta, \gamma, \eta, \rho$,

$$\cos \lambda x \tan ux = \sin \lambda x, \qquad \sin \lambda x \cot ux = \cos \lambda x,$$

and hence as before,

$$(27)\quad \varphi^{2n}(\lambda, 2Gu) = -4u\lambda_u^{2n}, \qquad \varphi^{2n+1}(\lambda, 2Bu) = 2u\lambda_u^{2n+1}.$$

These express the functions of given rank linearly in terms of functions of the same kind and lower ranks. To generalize we proceed as in § 2, getting

$$(28)\quad \begin{aligned} &f(x+\lambda+2Gu)+f(x-\lambda-2Gu)\\ &\qquad +f(x+\lambda-2Gu)+f(x-\lambda+2Gu)\\ &\qquad\qquad = -4u[f_u(x+\lambda)+f_u(x-\lambda)], \end{aligned}$$

$$(29)\quad \begin{aligned} &f(x+\lambda+2Bu)-f(x-\lambda-2Bu)\\ &\qquad +f(x+\lambda-2Bu)-f(x-\lambda+2Bu)\\ &\qquad\qquad = 2u[f_u(x+\lambda)-f_u(x-\lambda)], \end{aligned}$$

which hold for $\lambda = \beta, \gamma, \eta, \rho$. The identities from which these are inferred may be recovered as indicated above.

A relation between functions f whose arguments contain at most r distinct umbræ chosen from among $\beta, \gamma, \eta, \rho$ is called r-fold. It would be of interest to write out a set of such relations for $r = 1, 2, 3, 4$ containing 4, 6, 4, 1 relations of the respective types (the case $r = 1$ is treated above); but to save space we indicate the method for $r > 1$ in one case only. When $r = 2$ the simplest relations are found by multiplying together two of the generating identities and proceeding as above. Thus (19), (20) give

$$\begin{aligned} \cos \psi\,(\eta, \rho)x &= -4\cos 2Rx + 2\cos 2\rho x,\\ \cos \varphi\,(\eta, \rho)x &= 4\cos 2Rx + 2\cos 2\rho x; \end{aligned}$$

and hence

$$(\eta \pm \rho)^{2n} = 2^{2n+1}\rho^{2n}, \quad 2^{2n+2}R^{2n},$$

the upper sign giving the first on the right, the lower the second. Generalizing we get

(30) $f(x+\eta+\rho)+f(x-\eta-\rho)=2f(x+2\rho)+2f(x-2\rho),$

(31) $f(x+\eta-\rho)+f(x-\eta+\rho)=4f(x+2R)+4f(x-2R).$

Other simple types arise from division; thus

$$f(x+\eta+2B)+f(x+\eta-2B)=4f(x+\rho).$$

5. *Functions of Argument* $2u$, $2u-1$. In many applications the values of the functions in § 3 are required only for the arguments $2u$, $2u-1$. It is then desirable to consider the functions of $2u$, $2u-1$ as new functions of u. Accordingly we write the definitions

$$\lambda_j^n(u) \equiv \lambda_{j,n}(u) = \lambda^n(2u-j), \qquad (j=0,1),$$

for $\lambda = \beta, \gamma, \eta, \rho$. There can be no confusion between the constant suffix j and the u in λ_{ju}^n denoting the u-derivative of λ_j^n (the argument u being omitted as before). We have $\lambda_j^n(u) = (\lambda(2u)-j)^n$, as is evident on expanding by Taylor's theorem and noting (3). The generating identities are written down from (17)–(20) by changing u into $2u-j$ and replacing λ by λ_j. Hence all the relations between the λ_j^n $(j=0,1)$ functions of either kind can be written down immediately from those for the λ^n functions by replacing in the latter relations λ by λ_j and u by $2u$ or $2u-1$ according as $j=0$ or 1. It is therefore unnecessary to repeat any of the derivations.

We note one relation of particular interest, the analogue for the $\lambda_1^n(u)$ $(\lambda=\beta,\gamma,\eta,\rho)$ of relations found by Raabe, Schlömilch and Glaisher for their functions, the obviousness of which by the symbolic method is striking:

(32) $$\lambda_1^n(u) = (-1)^n\lambda_1^n(1-u), \quad (\lambda=\beta,\gamma,\eta,\rho).$$

Another type may be noticed. Expand $\sin(2u-1)x$, $\cos(2u-1)x$ in the generating relations by the addition theorems for the sine and cosine, use (17), (18) to reduce the results, and finally equate coefficients of like powers of x. Thus

$$\begin{aligned}\sin\beta_1x &= 2x\operatorname{ctn}x\,[\sin 2ux\cos x-\cos 2ux\sin x]\\ &= 2x\csc x\sin 2ux-2x\cos(2u-1)x,\\ &= 2\sin\rho(2u)x-2x\cos(2u-1)x,\end{aligned}$$

and similarly for $\cos\beta_1x$ and the rest. We find hence

(33) $$\beta_1{}^n(u) = 2\rho^n(2u) - 2n(2u-1)^{n-1},$$

(34) $$\gamma_1{}^n(u) = -2n\eta^{n-1}(2u) + 4n(2u-1)^{n-1},$$

(35) $$2n\eta_1{}^{n-1}(u) = \gamma^n(2u) + 4n(2u)^{n-1},$$

(36) $$2\rho_1{}^n(u) = \beta^n(2u) - 2n(2u)^{n-1};$$

and generalizing,

(37) $$f(x+\beta_1) = 2f(x+\rho_0) - f_u(x+2u-1),$$

(38) $$f(x+\gamma_1) = -f_u(x+\eta_0) + 2f_u(x+2u-1),$$

(39) $$f_u(x+\eta_1) = f(x+\gamma_0) + 2f_u(x+2u),$$

(40) $$2f(x+\rho_1) = f(x+\beta_0) - f_u(x+2u).$$

Thus if a relation between any members of either set λ_j ($\lambda = \beta, \gamma, \eta, \rho, j = 0, 1$) be given, a relation between the corresponding members of the other set can be written down from it, β, ρ and γ, η being the pairs of correspondents.

6. *Values of the* $\lambda_j{}^n$ ($j = 0, 1$) *for Integral Arguments* > 0. The connection between our functions and those of other writers is obtained (among other ways) by noting the sums of like powers of integers to which the $\lambda_j{}^n$ degenerate when the argument u is an integer > 0. There are 16 cases in sets of four each, obtained by multiplying $\sin 2ux$, $\cos 2ux$, $\sin (2u-1)x$, $\cos (2u-1)x$ by $x \operatorname{ctn} x$, $2x \tan x$, $\sec x$, $x \csc x$, reducing the products to linear functions of circular functions (of which not more than one in any case is other than a sine or cosine), and taking the coefficient of $x^{2n}/(2n)!$ or $x^{2n+1}/(2n+1)!$ according as the expanded product is an even or an odd function of x. These coefficients are simply expressible in terms of λ_j ($j = 0, 1$) functions; they also are readily expressed in terms of the Bernoullian and Eulerian functions already in the literature. A full discussion of them having been given elsewhere it suffices here to state the reference.*

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* MESSENGER OF MATHEMATICS, vol. 50 (1921), pp. 177–186. The functions φ, ψ of the paper differ slightly from those of the present, also β, γ there have no connection with β, γ here.

ON THE EXISTENCE OF CURVES WITH ASSIGNED SINGULARITIES *

BY J. L. COOLIDGE

The fundamental relations which connect the point and line singularities of algebraic plane curves are called Plücker's equations; they give necessary limitations upon the numbers of characteristics of the different sorts. These equations do not, however, contain in themselves any existence theorems, and after a solution in integers has been obtained, there is no guarantee that there is any curve whose characteristics are the numbers found. Erroneous views have been held on this point. The fact that a quartic is possible with three cusps suggests the existence of a unicursal curve of any order all of whose singular points are cusps; such a curve can not exist.

Various attempts have been made to show the existence of curves with assigned singularities, the most important being that of Lefschetz.† As far as simple nodes are concerned, he completed the solution, exhibiting the existence of plane curves with no singularities but simple nodes, and these in any desired number up to the theoretical maximum. He also showed that the requirement of each additional node imposed just one new condition. With regard to cusps, he was less successful. He established certain upper limits which may be attained, but the conclusions are not clean-cut, and depend upon what he calls the *Postulate of singularities*, which consists essentially in assuming that when we require a certain curve to have an additional cusp, we do not thereby impose upon it automatically more than one additional cusp.

The present paper follows closely Lefschetz's methods, but reaches a more definite and much simpler conclusion, and does so without the use of his postulate, which is proved in the course of the work. The final result is as follows.

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* Presented to the Society, December 27, 1922.

† *On the existence of loci with given singularities*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), p. 23.

THEOREM. *The maximum possible number of cusps for a curve of odd order n and deficiency p, where* $n \geqq 2p + \sqrt{8p+9}$, *is*

$$\frac{3}{2}[(n-2)+2p]-\frac{1}{2}\cdot$$

The maximum number for a curve of even order n and deficiency p, where $n \geqq 2p + 1 + \sqrt{8p+1}$, *is*

$$\frac{3}{2}[(n-2)+2p]\cdot$$

There exist curves of order n and deficiency p having cusps in any number up to the maximum.

Let us begin by writing some of Plücker's numbers, and the equations which connect them:

Order of curve	$= n,$	Number of cusps	$= \kappa,$
Class of curve	$= m,$	Deficiency	$= p,$
Number of nodes	$= \delta,$	Number of inflections	$= i.$

We have, then, the following equations of Plücker:

$$m = n(n-1) - 2(\delta+\kappa) - \kappa, \tag{1}$$

$$p = \frac{(n-1)(n-2)}{2} - (\delta+\kappa), \tag{2}$$

$$\begin{aligned} m &= 2(n-1) + 2p - \kappa, \\ n &= 2(m-1) + 2p - i, \\ i &= 3n(n-2) - 6(\delta+\kappa) - 2\kappa, \\ i &= 3[(n-2)+2p] - 2\kappa, \end{aligned} \tag{3}$$

$$\kappa = \frac{3}{2}[(n-2)+2p] - \frac{i}{2}\cdot \tag{4}$$

In order to prove that a curve may have the maximum number of cusps we must show that we can have a curve of given order and deficiency with one inflection, if the order be odd, and no inflection if it be even.

Suppose that we have a unicursal curve of order n with a multiple point A of order $n-1$, all the tangents being distinct. Let B be a simple point of this curve. Pass a quadric surface through A and B, and let V be the point where a generator through A meets a generator through B. Let us project our plane curve upon this quadric from V. The space curve will not pass through V, but will meet VA at the points of contact of the tangent planes through the tangents to the plane curve at A, and will meet VB in the point of contact of the tangent plane through the tangent at B to the plane curve. Since the space curve does not go through V, and meets a plane through V in n points other than V, it is a space curve of order n. Now if we project this space curve back upon the plane from a point which does not lie on one of its tangents, or on the common secant of two intersecting tangents, we get a plane curve, also of order n, whose only singularities are nodes, and this plane curve is unicursal, for it is birationally related to the first one. But if the point of projection lie upon a tangent to the space curve the new plane curve will have just one cusp, but a deficiency 0 as before. We see, therefore, that we may have a unicursal curve with no singularities but nodes,* or with a single cusp. The classes in these two cases are, by (3),

$$2(n-1); \qquad 2(n-1)-1.$$

If we increase n to $n+1$, we shall get the classes

$$2n; \qquad 2n-1.$$

Thus it appears that we can find a unicursal curve with one cusp and any odd class above 1, or with no cusp and any even class. Dually, we can find a curve which is unicursal with one inflection and any odd order, or with no inflection, and any even order, and this shows that we can find a unicursal curve with the maximum number of cusps. Now for a curve of deficiency p, we follow a general form of reasoning of Lefschetz.

There will exist a curve of deficiency p and the maximum number of cusps if there exist a curve of given order and deficiency with one inflection or no inflection, as the case may be, i.e., if there exist a curve of given deficiency and class with one cusp or no cusp, or, finally, a curve of given order and deficiency with one cusp or no cusp. Since the number of double tangents cannot be negative, when m and n are even, $i = 0$, and

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* See Snyder, *Construction of curves of given deficiency*, this BULLETIN, vol. 15 (1908), p. 1.

$$(m-1)(m-2) > 2p, \qquad 2(m-1) \geqq 1 + \sqrt{8p+1},$$

$$n \geqq 2p + 1 + \sqrt{8p+1};$$

when n is odd, $i = 1$, and

$$n \geqq 2p + \sqrt{8p+9}.$$

What is the contrary hypothesis? It is that if a curve have deficiency p and given order it must have more than one cusp or the equivalent in higher singularities. If this were true for every curve of deficiency p it would be true for every curve of lesser deficiency, for the curves of lesser deficiency have more singularities, i.e., they have imposed upon them all the conditions for the curves of higher deficiency, and others also. But this last hypothesis does not hold where the deficiency is 0; hence it does not hold in the general case. Thus, there are curves of any deficiency with one cusp or no cusps, or with one inflection or no inflections, hence curves of any deficiency with the maximum number of cusps.

We have now, lastly, to prove that we can have any number of cusps less than the maximum. We can have no cusps at all or one cusp. Start with this and impose conditions for one cusp, then another, etc. We shall thus get any number of cusps up to the maximum unless it happen that at some stage of the process the imposition of an additional cuspidal condition will produce several new cusps. If this should happen, the intervening number of cusps would be impossible. But another result would be reached also, for the number of additional conditions imposed in order to change from a curve with no cusps or one cusp to one with the maximum number of cusps would be less than the number of cusps so gained.

If a curve have the order n and κ cusps, where this number is one or zero, according as n is odd or even, the number of degrees of freedom is

$$\frac{n(n+3)}{2} - \left[\frac{(n-1)(n-2)}{2} - p\right] - \kappa = 3n + p - 1 - \kappa.$$

Hence the number of degrees of freedom of a curve of class m with one inflection or no inflections is

$$3m + p - 1 - i,$$

where $i = 0$ or $i = 1$.

If the conditions for cusps are all independent, and only in that case, the amount of freedom would be

$$\frac{n(n+3)}{2} - \left[\frac{(n-1)(n-2)}{2} - p\right] - \frac{3}{2}[(n-2) + 2p] + \frac{i}{2} = \frac{3n}{2} + 2 - 2p + \frac{i}{2}\cdot$$

But

$$m = 2(n-1) + 2p - \frac{3}{2}[(n-2) + 2p] + \frac{i}{2} = \frac{n}{2} + 1 - p + \frac{i}{2}\cdot$$

Hence

$$\frac{3n}{2} + 2 - 2p + \frac{i}{2} = 3m + p - 1 - i.$$

It appears, then, that a curve with the maximum number of cusps has exactly the freedom compatible with the appearance of one new cusp for each additional cuspidal condition, and with no other hypothesis. This proves Lefschetz' postulate of singularities, and shows that we may assign any number of cusps up to the maximum, without fear that any case is impossible, i.e., proves our theorem in its entirety.

HARVARD UNIVERSITY

A GENERALIZATION OF NORMAL CONGRUENCES OF CIRCLES*

BY J. L. WALSH

1. *Introduction.* A congruence of circles in three-dimensional space is said to be *normal* if every circle of the congruence is normal to three surfaces. Normal congruences have long been studied,† and one of their principal properties is expressed in the theorem that if a variable circle C is normal to the three fixed surfaces S_1, S_2, S_3 at the points P_1, P_2, P_3 respectively, and if the point P_4 is determined by a real constant cross ratio with P_1, P_2, P_3, then as C varies the point P_4 traces a surface which is also orthogonal to C.

It is the purpose of the present note to consider a type of congruence to which we shall give the name of isogonal congruence and which is a generalization of the notion of normal congruence. A congruence of circles is said to be *isogonal* if every circle of the congruence cuts three surfaces at equal angles in such a way that when the circle is inverted into a straight line L, the tangent planes to the corresponding surfaces at their points of intersection with L are all parallel. That is, every sphere through a circle of the congruence cuts at equal angles the three surfaces at their points of intersection with that circle. It is to be noted that the term isogonal might well be given to a still larger type of congruence of circles, but in the present paper the term will be used only in the restricted sense indicated.

We shall prove (Theorem III) that if a congruence is isogonal there are not merely three surfaces but a one-parameter family of surfaces which have the isogonal property, and all the surfaces of the family can be obtained as in the case of normal congruences.

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* Presented to the Society, December 27, 1922.

† By Ribaucour, Darboux, Bianchi, Eisenhart, and Coolidge, among others. Detailed references are given by Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, Chap. XV.

Isogonal congruences of circles are thus a generalization of normal congruences of circles, of normal congruences of lines, and of certain congruences of lines which naturally arise in connection with the parallel mapping of surfaces*. Isogonal congruences are particularly interesting because in general three arbitrary surfaces determine one or several such congruences isogonal to them. If one fixed circle C is isogonal to three surfaces, there is in general one and only one congruence of circles isogonal to those surfaces and containing C. For the condition of isogonality for a circle C' and three surfaces C_1, C_2, C_3 is satisfied if two spheres through the circle C' cut at equal angles the surfaces C_1, C_2, C_3 at their points of intersection with C'. This is equivalent to four independent conditions on all the six-parameter family of circles in space.

Let us proceed to investigate the analogue in the plane of the isogonal congruence.

2. *Isogonal Series in the Plane.* The name *isogonal series* shall be given to a one-parameter family of circles in the plane such that each circle of the family cuts at equal angles three curves, and in such a way that when the circle is inverted into a straight line, the tangents to the transformed curves at their points of intersection with the transformed circle are all parallel. In general, three arbitrary curves determine one or more isogonal series, and if a circle C cuts isogonally three curves there is in general one and only one isogonal series which contains C and all of whose circles cut isogonally those three curves. For the condition of isogonality to three curves involves two independent conditions on the three-parameter family of all circles of the plane. We shall proceed to prove the following theorem.

THEOREM I. *Let a variable circle C cut three fixed curves C_1, C_2, C_3 isogonally at the variable points z_1, z_2, z_3 respectively. Then the point z_4 defined by the real constant cross ratio*

$$(1) \qquad \lambda = (z_1, z_2, z_3, z_4) \equiv \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

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*Isogonal congruences are also a generalization of a type of congruences of circles considered by F. W. Beal, ANNALS OF MATHEMATICS, (2), vol. 17 (1915–16), pp. 180–170.

traces a curve C_4 such that the circle C cuts isogonally C_1, C_2, C_3, C_4 at the points z_1, z_2, z_3, z_4.

We fix our attention on a particular circle C and its points of intersection z_1', z_2', z_3' (supposed distinct) with C_1, C_2, C_3. We shall prove that *if z_1, z_2, z_3 move simultaneously from z_1', z_2', z_3' along C_1, C_2, C_3 in any way whatever, then the point z_4 defined by* (1) *traces a curve which C cuts isogonally with C_1, C_2, C_3.*

If z_1 moves from z_1' along C_1, but z_2 and z_3 are kept coincident with z_2' and z_3', then z_4 moves from z_4' along a curve which is cut by C isogonally with C_1. This becomes obvious if z_3' is transformed to infinity; equation (1) then represents a transformation (z_1, z_4) of similitude with z_2' as center, while C is a straight line which is unchanged by the transformation. Likewise if z_2 moves from z_2' along C_2, but z_1 and z_3 are kept coincident with z_1' and z_3', then z_4 moves from z_4' along a curve which is cut by C isogonally with C_2. A similar fact obtains if z_3 moves from z_3' along C_3. Thus, independent infinitesimal changes of z_1, z_2, z_3 from z_1', z_2', z_3' along C_1', C_2', C_3' move z_4 from z_4' along a curve of the kind described, so simultaneous infinitesimal changes of z_1, z_2, z_3 from z_1', z_2', z_3' along C_1', C_2', C_3' also move z_4 from z_4' along a curve C_4 such that C cuts the curves C_1, C_2, C_3, C_4 isogonally at z_1', z_2', z_3', z_4'.* This completes the proof of Theorem I.

Theorem I is particularly interesting in the case that C_1,

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* The detailed analysis of the differentials involved is extremely simple in the present case. We have by differentiation of (1) and substitution of (1) in the result,

$$(1') \quad dz_4 = \frac{(z_1 - z_2)dz_3 + (z_3 - z_4)(dz_1 - dz_2) - \lambda(z_2 - z_3)dz_1 - \lambda(z_4 - z_1)(dz_2 - dz_3)}{(z_1 - z_2) + \lambda(z_2 - z_3)}.$$

Transform the circle C of Theorem I into the axis of reals, identify the real values z_i' of Theorem I with the z_i of (1′), and identify the variables z_i of Theorem I with the $z_i + dz_i$ of (1′). All the quantities in (1′) except the differentials are real; all the differentials of the right-hand member have the same argument (mod π), so dz_4 has also that same argument (mod π).

Thus if $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$ are solutions of a Riccati equation whose cross ratio is a real constant, then whenever the $z_i(t)$ vary as functions of t, they trace paths that are cut isogonally by the circle on which the $z_i(t)$ lie.

C_2, C_3 are all circles. In this case we have the following theorem.*

THEOREM II. *Let C_1, C_2, C_3 be three fixed non-coaxial circles. Then the circles C which cut isogonally C_1, C_2, C_3 at points z_1, z_2, z_3 form four distinct series,† each of which is composed of the circles of a coaxial family. If there are considered the circles C of but one isogonal series, the point z_4 defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

traces a circle C_4 which is such that C cuts isogonally C_1, C_2, C_3, C_4 at z_1, z_2, z_3, z_4.

If C_1, C_2, C_3 are coaxial but not all tangent at a single point, there is but one series of circles C cutting them isogonally, namely the circles of the coaxial family conjugate to the family to which C_1, C_2, C_3 belong, and all these circles C cut the three given circles orthogonally. But the points z_1, z_2, z_3 may be chosen on C and on the circles C_1, C_2, C_3 in four essentially different ways, always so that C cuts C_1, C_2, C_3 isogonally at z_1, z_2, z_3. Thus we still have four circles C_4 and for any particular choice of C_4, the circle C cuts isogonally C_1, C_2, C_3, C_4 at z_1, z_2, z_3, z_4.

If C_1, C_2, C_3 are all tangent at a single point P, any circle C through P cuts the original circles isogonally at the intersections of C and those circles distinct from P, so no isogonal series is defined. We can still obtain the four circles C_4, however, by requiring respectively (1) that no point z_1, z_2, z_3 shall coincide with P; (2) that z_1 shall always lie at P; C must then be orthogonal to C_1, C_2, C_3; (3), (4) similarly for z_2 and z_3. Always the circle C_4 is traced by the point z_4, and the circle C cuts C_1, C_2, C_3, C_4 isogonally at z_1, z_2, z_3, z_4.

3. *Isogonal Congruences in Space.* Theorem I and its proof

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* See Walsh, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 101–116; Lemma IV.

† By a proper convention for the *angle between two circles*, these four systems are described respectively by saying that C cuts C_1, C_2, C_3 all at the same angle or one of those circles at an angle supplementary to the angle cut on the other two. A similar remark obtains below for Theorem IV.

as just given extend directly to space. Let us prove the following theorem.

THEOREM III. *Let a variable circle C cut isogonally three fixed surfaces C_1, C_2, C_3 at the variable points P_1, P_2, P_3. Then the point P_4 defined by the real constant cross ratio*

$$\lambda = (P_1, P_2, P_3, P_4) \tag{2}$$

traces a surface C_4 such that the circle C cuts isogonally C_1, C_2, C_3, C_4 at the points P_1, P_2, P_3, P_4.

We fix our attention on a particular circle C and its points of intersection P_1', P_2', P_3' (supposed distinct) with C_1, C_2, C_3. We shall prove that *as P_1, P_2, P_3 move from P_1', P_2', P_3' on C_1, C_2, C_3 in any way whatever, then P_4 as defined by* (2) *traces a surface C_4' such that C cuts isogonally C_1, C_2, C_3, C_4' at P_1', P_2', P_3', P_4', where P_4' is defined by $\lambda = (P_1', P_2', P_3', P_4')$.*

If P_1 moves from P_1' along C_1, but P_2 and P_3 are kept coincident with P_2' and P_3', then P_4 traces a surface which is cut by C at P_4' isogonally with C_1, C_2, C_3 at P_1', P_2', P_3'. The corresponding fact holds if P_2 or P_3 is allowed to move on C_2 or C_3 while the other two of the original three points are kept fixed. Independent infinitesimal changes of P_1, P_2, P_3 from P_1', P_2', P_3' along C_1, C_2, C_3 therefore move P_4 along a surface of the kind described, so simultaneous infinitesimal changes of these points must move P_4 along a surface C_4' such that C cuts isogonally C_1, C_2, C_3, C_4' at P_1', P_2', P_3', P_4'. Thus even if a congruence is not isogonal but a single circle C of the congruence cuts isogonally the surfaces C_1, C_2, C_3, then C cuts isogonally with C_1, C_2, C_3 the surface traced by the point P_4 defined by (2).

We leave to the reader the proof of the following theorem, which is the space analogue of Theorem II.

THEOREM IV. *Let C_1, C_2, C_3 be three fixed non-coaxial spheres. Then the circles C which cut isogonally C_1, C_2, C_3 at points P_1, P_2, P_3 form four distinct congruences, each of which is composed of the circles through two points, real, coincident, or imaginary. If there are considered the circles C of but one isogonal congruence, the point P_4 defined by the real constant* *cross ratio*

$$\lambda = (P_1, P_2, P_3, P_4)$$

traces a sphere C_4 *which is such that* C *cuts isogonally* C_1, C_2, C_3, C_4 *at* P_1, P_2, P_3, P_4.

If the spheres C_1, C_2, C_3 are coaxial but not all tangent at a single point, there is but one congruence of circles cutting them isogonally; all of these circles C cut C_1, C_2, C_3 orthogonally. However, the points P_1, P_2, P_3 may be chosen on C and on their proper spheres in four essentially different ways, and in each case C cuts isogonally C_1, C_2, C_3 at P_1, P_2, P_3. Thus the point P_4 still traces four spheres C_4; and for any particular C_4, the circle C cuts isogonally C_1, C_2, C_3, C_4 at P_1, P_2, P_3, P_4.

If the spheres C_1, C_2, C_3 are all tangent at a single point P, any circle C through P cuts those spheres isogonally at the intersections of C and these spheres distinct from P, so we have no unique isogonal congruence. There are, however, four spheres C_4 which can be obtained by requiring respectively that P_1, P_2, P_3, or that none of those points should coincide with P. In the former cases the circle C must be orthogonal to C_1, C_2, C_3 to have the proper isogonal property; in the latter case all the circles C to be considered form a complex instead of a congruence. In every case the point P_4 traces a sphere C_4 such that C_1, C_2, C_3, C_4 are cut isogonally by C at P_1, P_2, P_3, P_4.

Theorems III and IV can be extended readily to any number of dimensions.

We add the remark that isogonal congruences arise in space naturally if we consider the problem of finding the locus of a point P_4 defined by the real constant cross ratio

$$\lambda = (P_1, P_2, P_3, P_4) \tag{3}$$

when the points P_1, P_2, P_3 have as their respective loci the regions R_1, R_2, R_3, or the surfaces S_1, S_2, S_3. The surfaces C_1, C_2, C_3 defining the isogonal congruence are the boundaries of the regions R_1, R_2, R_3 or the surfaces S_1, S_2, S_3 themselves. The boundary of the locus of P_4 is traced by P_4 as defined by (3) when the circle C of the congruence cuts C_1, C_2, C_3 isogonally at P_1, P_2, P_3. Detailed consideration of the corresponding fact for the plane has been given in a paper by the writer,* and can easily be extended to space by the reader.

HARVARD UNIVERSITY

A CORRECTION

BY EINAR HILLE

In the June number of this BULLETIN (vol. 28, No. 5, p. 261), the author published a paper with the title *Convex distribution of the zeros of Sturm-Liouville functions.* Through an oversight the last paragraph of the paper is inaccurate. We list the necessary corrections below.

Page 264, lines 4–10: Instead of "Note the lineal . . . φ_0.", read "On l we mark the eventual points a_n as well as the points where either $\arg G(z) = \arg G(z_1) + \pi$ or $\theta_z \equiv \varphi_0 \pmod \pi$. Let $z_2 = z_2(\varphi_0)$ be the first of these points, different from z_1, which we encounter when proceeding along the ray, the rest of which we leave out."

Page 264, line 13: Instead of "an analytic curve", read "either of two analytic curves, namely $A(z_1)$ which is the locus $\arg G(z) = \arg G(z_1) + \pi$, and".

Page 264, lines 24–27: Instead of "$l(\varphi_1)$ . . . respectively", read "$l(\varphi_1)$, considered as a double ray if necessary, from z_2^- to z_2 and from z_2 to z_2^+, we make the boundary curve continuous at $\varphi = \varphi_1$".

Page 264, line 28: Instead of "cuts", read "straight lines".

Page 264, line 30: After "the part of", insert "$A(z_1)$ and".

Page 265, first line: Leave out "on the cuts".

Same page, lines 6–10: Replace "Then . . . depends upon z_3" by "Then we can find an angle ϑ such that the two inequalities

$$\left\{\begin{array}{l} \vartheta < \Theta < \vartheta + \pi; \\ 2k\pi < \Theta < 2(k+1)\pi; \end{array}\right. \qquad \Theta = \arg\left[G(z)(z - z_1)^2\right], \tag{15}$$

will hold for all interior points on the segment (z_1, z_3), where k is some integer".

PRINCETON UNIVERSITY

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* See Lemma III of the paper to which reference has already been made, and also TRANSACTIONS OF THIS SOCIETY, vol. 23 (1922), pp. 67–88, Theorem II.

SHORTER NOTICES

Vorlesungen über Zahlen- und Funktionenlehre. Erster Band, dritte Abteilung. By Alfred Pringsheim. Leipzig and Berlin, B. G. Teubner, 1921. ix + 515–976 pp.

The first part of this volume was reviewed in a previous number of this BULLETIN (vol. 25 (1919), p. 470). The second part, which appeared shortly after the first, was devoted to a detailed exposition of the theory of infinite series with real terms. The third part treats of complex numbers, infinite series with complex terms, infinite products, and continued fractions. It also contains an appendix to the whole first volume and an index. The appendix (pp. 917–969) consists of numerous references to the literature and indications of the historical development of portions of the theory, as well as further discussion of many of the topics treated in the body of the text.

The exposition exhibits Professor Pringsheim's characteristic lucidity, and the theory throughout is developed from first principles in elementary fashion. By the phrase "elementary fashion" we do not wish to connote anything in the nature of looseness of treatment or lack of rigor, for the author has been at particular pains to avoid defects of that sort. Thus the book is well adapted to meet the needs of any reader, regardless of the extent of his previous mathematical knowledge, who wishes to have a complete and logically accurate account of the arithmetic foundations of modern analysis.

In spite of the elementary treatment, the book is in no way limited to the elements of the subject and in connection with various particular topics comes close to the confines of our present-day mathematical knowledge. For example in the second chapter of Section III,* as a preparation for the treatment of the multiplication of series, the methods of Hölder and Cesàro for summing divergent series are introduced and the equivalence of these two methods, a theorem of comparatively recent date, is established. In this connection it is of interest to note that Professor Pringsheim objects to the use of the phraseology "summation of a divergent series" and introduces in place of it the expression "reduction (Reduktion) of a divergent series." He thus speaks of a certain series as being reducible (reduzibel) with a particular associated limit (zugeordneter Grenzwert). He bases his objection to the current usage on the statement that at the present time there exists no precise definition of the term "sum of a divergent series," and in his opinion an epithet so expressive and sonorous (ein so prägnant klingendes Beiwort) as summable should have a precise and definite meaning, such as the words convergent and divergent possess.

It will be recalled by those particularly interested in the study of divergent series that the objections raised by Professor Pringsheim to the present use of the word summable are quite similar to those previously expressed by Professor W. B. Ford in his book *Studies on Divergent*

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* The whole first volume has been divided into four principal sections, of which Part three contains Sections III and IV.

Series and Summability and his address entitled *A conspectus of the modern theory of divergent series*, published in volume 25 of this BULLETIN. It will also be recalled that the remedy proposed by the latter was more drastic, namely to give the word summable a definite meaning, based on the notion of analytic extension, and thus exclude from consideration any divergent series that did not satisfy the proposed definition of summability. A combination of the suggestions of Professors Ford and Pringsheim, namely the use of Professor Ford's definition to set off a definite class of divergent series to be known as summable series, and the use of Professor Pringsheim's term reducible in connection with other divergent series might be more generally acceptable than either suggestion alone. It is the opinion of the reviewer, however, that the present use of the word summable, even if it does involve a certain vagueness in the meaning of such an excellent word, is rather too firmly entrenched to be changed overnight. But the objections of Professors Ford and Pringsheim do have a certain validity, and regardless of the opinion of particular individuals the final terminology will probably be an illustration of the survival of the fittest.

In connection with the introduction of complex numbers in Chapter I of Section III, the author departs from the generally adopted method of first building up a system of pairs of real numbers with arbitrarily assigned rules of combination. He begins by introducing the pure imaginary as a new type of number adapted to provide a solution for equations of the type $x^2 = -a^2$ (a real). Then after developing the properties of this new system on the basis of the principle that in their rules of combination they should obey the fundamental laws of algebra, he leads naturally to the introduction of complex numbers by showing that the result of adding a real number and an imaginary number will not be a number of either of these types.

The treatment of continued fractions, to which all of Section IV is devoted, is particularly complete. As regards questions of convergence it contains practically all of our present-day knowledge of the subject. In his preface the author expresses the hope that this part of the book will serve to gain for this highly important, and in his opinion not sufficiently appreciated, topic new devotees from the ranks of mathematical novitiates.

The book proper contains relatively few references to the literature, these having been in the main relegated to the appendix. This arrangement is in line with the purpose of making the treatment particularly suitable for beginners, since it avoids undue interruption in the orderly presentation of the theory and the possibly forbidding aspect of numerous footnotes and citations in the body of the text. With regard to the appendix the author specifically disclaims any pretensions to completeness and excuses himself in advance in case he may on this account have passed over priority claims of particular individuals. In this connection he expresses the opinion that the custom of attaching the names of the discoverers to mathematical theorems has been carried too far in recent times and has led to what he considers a depreciation of the value of mathematical immortality. For this reason he has in general avoided the use of the name of the originator in connection with a theorem or a method except in instances where the usage may be regarded as classical. One is tempted to inquire, however, how these classical usages could have originated if some one had not initially honored an author by naming one of his discoveries after him? It would seem that the whole question reduces to a matter of taste, as to whether one prefers the personal or the impersonal point of view in the study of mathematical science.

C. N. MOORE

Theorie des Potentials und der Kugelfunktionen. By A. Wangerin. Band II, Sammlung Schubert LIX. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. viii + 286 pp.

The first volume of the above work, which deals with the fundamental portions of the potential theory, was reviewed in a previous number of the BULLETIN (vol. 16 (1910), p. 492). The present volume is devoted to a study of the properties of spherical harmonics and their applications to various problems of potential theory.

The book is divided into four sections. The first section deals with such properties of spherical harmonics as are essential for later developments; the second deals with potential problems for the sphere; the third with potential problems for the ellipsoid of revolution, for two spheres, and for a few other special cases; the fourth with potential problems for arbitrary closed surfaces.

The book as a whole contains more material than one would expect to find in a work of its size, and the presentation is in general clear and sufficiently refined for the purpose in hand. There are, however, certain errors and inconsistencies in the statements and demonstrations of some of the fundamental theorems that should be eliminated. For example, any one familiar with the literature on spherical harmonics will be surprised to find on page 93 the claim that the author has established the convergence of the development in Laplace's functions of any function $f(\theta, \varphi)$ that is finite, single-valued, and continuous on the whole sphere. Since examples have been constructed by Haar and Gronwall of continuous functions whose development in spherical harmonics is divergent, one naturally examines the "proof" of the so-called theorem with some curiosity. On page 89 he finds the assumption that a sum of p terms, each one of which approaches zero, will also approach zero, even though p becomes infinite at the same time that the individual terms approach zero. On page 93 he finds a statement that is equivalent to the assumption that a function continuous throughout a certain interval does not have an infinite number of maxima and minima in that interval.

It would surely be better to omit entirely any proof of the development theorem, important as it is for the subsequent theory, than to introduce such obvious errors with regard to the fundamentals of analysis in the course of the demonstration. A less radical remedy is available, however, for by adding a further restrictive condition to the statement of the theorem and a relatively small amount of material to the proof, the whole discussion could be put on an entirely rigorous basis. It is to be hoped that these changes will be made, if a subsequent edition appears.

C. N. MOORE

Lezioni sulla Teoria dei Numeri algebrici e Principi d'Aritmetica analitica. By Luigi Bianchi. Pisa, Enrico Spoerri, 1921. viii + 444 pp.

This work consists of a 50-page introduction and three chapters. The introduction contains a development of the theory of numbers for the field $k(i)$. The theorems of the theory of the rational numbers are extended to this field giving the proof of the unique separation into prime factors, the development of the φ function, the proof of Fermat's theorem, the theory of binomial congruences with proof of the existence of primitive roots of the congruence $x^{\varphi(p)} \equiv 1 \bmod p$, and the theory of indices and quadratic residues with the law of reciprocity.

The first and second chapters are devoted to the theory of algebraic numbers in a general field of degree n. The first chapter contains the development of that part of the theory which is independent of factorization. The existence of a base is shown and the discriminant of a field is defined. The greater part of the chapter is devoted to the theory of units which is developed from Minkowski's theorem on linear forms leading up to the proof of the existence of a fundamental system of units.

The second chapter contains the theory of ideals, showing the purpose of their introduction into the theory of algebraic numbers, proving the unique factorization theorem, and discussing congruences with respect to an ideal modulus. The chapter also contains the theory of the classification of ideals, the discussion of the group of classes, and the correspondence between the classes of ideals and equivalent decomposable forms.

The last part of the chapter is given over to a brief consideration of the theory of ideals in Galois, Abelian, cyclotomic, and Kummer fields. The law governing the factorization of the rational primes in a cyclotomic field of prime degree is developed. This is also done earlier in the chapter for quadratic fields as an application of the theory of quadratic residues with respect to an ideal modulus.

The last chapter contains a good presentation of the analytic theory of ideals. The Riemann and Dedekind zeta functions are studied showing the relation of the latter to the problem of the determination of the number of classes in a field.

The zeta functions for quadratic and cyclotomic fields are reduced to their simpler forms and the method for the determination of the number of classes in a quadratic field is given for the case when the discriminant of the field is congruent to 1 mod 4.

The last paragraph contains a brief indication of Hecke's results regarding the zeta function.

G. E. WAHLIN

Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. By Dr. Edmund Landau. Leipzig and Berlin, B. G. Teubner, 1918. viii + 143 pp.

This work is divided in two parts. The first part (54 pages) contains a brief, but unusually clear, presentation of the elementary theory of algebraic numbers and ideals in an algebraic number field.

The author states in the preface that the aim of the work is not to give an exhaustive treatment, but to bring the reader up to and a little beyond the fundamental theorem of Dedekind that the separation of an ideal into the product of prime ideals is unique.

The subjects considered beyond the work necessary for the proof of the theorem mentioned are: the theory of the classification of ideals showing that the number of classes is finite, and the theory of units in an algebraic number field, with the proof of the existence of a fundamental system of units.

The second part is devoted to the analytic theory of ideals. The first chapter gives a clear introduction to the functions $\zeta(s)$ and $\zeta(s; \kappa)$, where κ is a class of the field. It is shown that, except for a pole of the first order in $s = 1$, the functions are regular in the entire plane, and that the residue of $\zeta(s; \kappa)$ in $s = 1$ is a number λ, independent of the class κ, while that of $\zeta(s)$ is λh, where h is the number of classes. Hecke's functional equations for $\zeta(s)$ and $\zeta(s; \kappa)$ are also developed.

The second chapter is a study of the distribution of the zeros of $\xi(s)$, and the third chapter leads to the proof of the author's remarkable theorem that, asymptotically, the number of prime ideals is the same in all fields.

The last chapter contains the result of the author's researches regarding the number of ideals, in a field or a class, whose norms are less than x. If this number be denoted by $H(x)$ for the field and $H(x; \kappa)$ for a class, it is shown that

$$H(x; \kappa) = \lambda x + O(x^{\vartheta})$$
$$H(x) = \lambda h x + O(x^{\vartheta})$$

where λ and h have the meaning given above,

$$\vartheta = 1 - \frac{2}{n+1}$$

and $O(x^{\vartheta})$ is a function whose quotient by x^{ϑ} is limited for sufficiently large x. It is shown that the exponent ϑ cannot be less than

$$\frac{1}{2} - \frac{1}{2n}\,.$$

The last four pages contain a brief historical survey with examples from a quadratic field, as well as notes of reference to the literature bearing on the various sections.

G. E. Wahlin

Lehrbuch der Funktionentheorie. By L. Bieberbach. Band I. *Elemente der Funktionentheorie.* Leipzig, Teubner, 1921. 6 + 314 pp.

Funktionentheorie. By L. Bieberbach. Teubner's Technische Leitfaden, Bd. 14. Leipzig, Teubner, 1922. 118 pp.

The author in his preface to the first book pleads guilty to entertaining the hope that he has written a text-book on complex function theory. He proceeds to set forth the qualities that such a text-book should have, viz: completeness, clarity, simplicity and unity. That the author succeeds in giving a clear, elementary presentation of the fundamental principles of the theory of functions of a complex variable cannot be gainsaid.

The table of contents indicates the usual order of topics.

1. Elementary treatment of complex numbers as number pairs, treated arithmetically and geometrically. 2. Limits and series, containing the fundamental concepts and definitions. 3. Continuity, region, derivatives, series of functions, conformal mapping. 4. The elementary functions of a complex variable. 5. Integration. 6. Cauchy's integral formula. 7. Theory of residues. 8. Analytic continuation. 9 and 10. Algebraic functions. 11. Elliptic functions. 12. Simple periodic functions. 13. Expansion of analytic functions in infinite series and products. 14. The Gamma function.

The printing is very good, unusually free from errors. The reviewer wishes to point out that on page 34 in the paragraph on the inverse function defined by $w = f(z)$ the requirement $f'(z_0) \neq 0$ should be added. Again on page 35 in the paragraph in which Laplace's differential equation is derived, the condition necessary to make the equation $v_{xy} = v_{yx}$ valid is not stated.

The eighty figures in the book are illuminating and helpful. Perhaps the chief contribution is contained in the use of the modern point-set theory in the fundamental definitions. This is a logical development of the fundamental importance which this theory is rapidly assuming in all branches of mathematics, especially in function theory. Two important theorems are to be noted, the first an extension of Weierstrass's theorems on a uniformly convergent series of analytic functions and an extension of Mittag-Leffler's theorem. The first, on page 165, is due to Vitali, and the second, on page 292, is due to Runge. The book as a whole is a welcome addition to complex function theory and marks a distinct contribution.

The second book is a simplified summary of the Lehrbuch and possesses the same merits as the larger text. The last two chapters contain applications to potential theory and hydrodynamics.

H. J. Ettlinger

Vorlesungen über Höhere Mathematik. By Hermann Rothe. Vienna, L. W. Seidel und Sohn, 1921. xi + 691 pp.

This book is strikingly like the first volume of Pierpont's *Theory of Functions of a Real Variable,* both in scope and in the spirit of rigor which pervades it. Its larger, more closely printed pages contain much more detail of proof, but beyond a considerable wealth of illustrative examples and some geometric applications of the calculus, the material and the degree of generality of the theorems are almost identical. At many points, the reader will find the proofs easier to study than those of Pierpont because every step is supplied. But at other points, he will find this same trait wearying and obscuring to the real essence of the demonstration. It is not likely that the author had any knowledge of the *Theory of Functions of a Real Variable,* for in supposing that he might be offering the first valid proof of the theorem on the derivative of a function of a function (see the preface), the only place in which he claims novelty of results, he overlooks the satisfactory proof given in that book.

The fact that these *Vorlesungen* are an elaboration of lectures given in a prescribed course in the Technische Hochschule at Vienna will mislead no one in this country into thinking that they can be of large service to technical students here. We appreciate rigor, to be sure, and feel the need of more of it. But we want also some analytical skill, we want some treatment of series, including Fourier series, and of approximation processes, and we want practice in the application of the calculus to problems of physics and engineering. And we do not care particularly for unusual generality at the expense of simplicity of proofs.

For the student of pure mathematics, on the other hand, the book has value. He may consult it with confidence, and will find it useful in supplying alternative proofs, or in supplying details of proofs given too succinctly elsewhere. He will find it well printed, well arranged, and supplied with figures which are models of clearness, if exception be made of a few, which, like the text, are overloaded with detail.

It is regrettable that so many mathematical texts are written without any clear purpose of stimulating self-activity. In the *Vorlesungen* there is not a single problem to be worked, and no detail of proof to be supplied. Many a mathematician owes his interest in the science to early bouts with problems, and the writer who gives careful attention to the selection of problems, or who includes as exercises valuable theoretical results, renders a high service. Both of these merits characterize, for instance, Appell's *Mécanique Rationnelle*, and Goursat's *Cours d'Analyse*. Had our present author followed some such course, he might have saved space, and produced a much more stimulating book.

O. D. KELLOGG

Plane Geometry. By L. B. Benny. Glasgow and Bombay, Blackie and Son, Ltd., 1922. vi + 336 pp.

The title of this book is misleading to American readers. The sub-title is as follows: "An account of the more elementary properties of conic sections treated by the methods of coordinate geometry and of modern projective geometry with applications to practical drawing." The contents will appeal to the American reader as a rather unusual mixture of elementary analytic geometry and projective geometry. In addition to the ordinary analytic treatment of the straight line and conic sections, the book contains chapters on ranges and pencils, harmonic properties of circles, inversion, projection, confocal conics, cross ratios and ends with a treatment of Pascal's and Brianchon's theorems. While the selection of contents is an unusual one in this country, the reader will find much of interest. Like most English texts, it is well supplied with exercises and problems of a type and difficulty unusual on this side of the Atlantic. The book also contains a number of portraits of mathematicians who have contributed to the subjects under discussion with an appendix giving biographical notes concerning them. These include Cayley, Riemann, Cremona, Descartes and Pascal.

J. W. YOUNG

Cours de Géométrie Analytique. By Georges Bouligand, with a preface by E. Cartan. Paris, Librairie Vuibert, 1919. vii + 421 pp.

This book constitutes a second course in the Cartesian geometry of the plane and of space, freely interspersed with essentials of modern geometry. After a concise development of rectangular and oblique coordinates and their application to straight lines and planes, the infinite and the imaginary elements are introduced. With the basis for projective geometry in complex space or for any of its sub-geometries thus developed, the author proceeds to consider some general questions. He first treats miscellaneous elementary properties of real curves and surfaces in real space. He then passes on to the discussion of projective and affine properties of algebraic curves and surfaces in complex space, developing the ideas of the order of a curve or a surface and that of one of its points, proving Bezout's theorem, and devoting some attention to asymptotes and to unicursal curves. After a brief discussion of loci comes the next major topic, that of transformations. Rigid motions, reflections, and transformations of similitude are discussed, followed by affine and projective transformations; cross ratio is introduced and the projective generation of conics and quadrics developed, involutions are given proper attention and Desargues' theorem is proved; finally comes the geometry of inversion.

An important phase of this chapter is the stress laid on the distinction between metric, affine, and projective geometries. Indeed, if the reviewer were to single out any one motive governing the plan of the book, it would be this motive. Latent but still decisive in the beginning, when once introduced it maintains predominance.

One further general concept, that of tangential coordinates, is developed, and the author then attacks the theory of curves and surfaces of the second order. With the general equation as the point of departure in each case, the usual questions of classification, normal forms, various geometric definitions (of the conics), rulings and circular sections (of the quadrics), tangential equations, poles and polars, diameters, etc., are discussed; a separate chapter treats geometrically the intersection of two quadrics.

A supplement of some sixty pages covers the geometrical applications of determinants, the classification of conics and quadrics on an analytic basis, the determination of conics and quadrics under given conditions, pencils of conics and quadrics, and an introduction to the theory of invariants.

The differential calculus is freely applied and a modicum of differential geometry developed. Certain subjects, whose detailed analytic treatments are regarded, perhaps unfortunately, by most American geometers as belonging to analysis, are carefully treated: the implicit function theorem in connection with a plane curve defined by an implicit equation, and a thorough analytic theory of envelopes in connection with line coordinates. On the other hand, the principle of duality is barely mentioned and, though polar reciprocation with respect to the sphere, $x^2 + y^2 + z^2 + 1 = 0$, is freely used, the theory of reciprocal polars in general is lightly passed over and that of general correlations not mentioned.

All in all, however, the author is to be congratulated on his choice of material; only essentially important topics are treated and these carefully and fully. There are no exercises per se, but the author leaves much in the text to the student and furnishes frequent suggestions for further study. The methods followed are for the most part analytic, but are inspired by a keen geometric insight. The proofs and deductions are concise, rigorous, facile, and in some cases, novel.

The author has followed an established custom in the descriptive, rather than logical, introduction of imaginary elements. This has led here, as frequently, to a lack of proper respect for the eccentricities of the complex domain, resulting in some unguarded, if not inaccurate, statements. It is to be hoped that the time is not far distant, when geometers who feel themselves forced into the complex domain will accord to it a logical and rigorous foundation.

The book makes delightful reading. It should prove of value to advanced students and to teachers of Cartesian and modern geometry in this country.

W. C. GRAUSTEIN

Applied Calculus. By F. F. P. Bisacre. Glasgow and Bombay, Blackie and Son, Ltd., 1921. xiv + 446 pp.

It is not in general the policy of the BULLETIN to review elementary textbooks on the calculus. In the present instance, however, it seems desirable to make an exception and to give a brief notice to Mr. Bisacre's text. It differs in several particulars from the usual elementary textbook. Its avowed intention is to provide an introductory course in the calculus for the use of students in the natural and applied science whose knowledge of mathematics is slight. The author does not presuppose anything beyond elementary algebra, trigonometry, geometry and graphs. In particular no previous course in analytic geometry is presupposed. The first part of the text as far as its selection of material is concerned does not differ very greatly from the ordinary textbook. It is, however, distinguished by a remarkable vividness of presentation. The author uses a quotation from the *Merchant of Venice,* for example, to illustrate the notion of units and large and small quantities, in particular the notion of allowable error, etc. In the latter part of the text, the selection of material, however, differs widely from the conventional. There is a chapter of thirty-two pages on problems in electricity and magnetism, a chapter of twenty-one pages on problems in chemical dynamics and a chapter of sixty-five pages on problems in thermodynamics. The treatment is remarkably accurate and rigorous considering its elementary character. Its interest is further enhanced by biographical notes and portraits of mathematicians who have contributed to the development of the calculus and to the physical theories covered.

It is doubtful whether the book could be advantageously used in courses on the calculus in this country and yet it is not at all clear that some schools of applied science or technology might not find it distinctly available in their work.

J. W. YOUNG

Die Grundgleichungen der Mechanik, insbesondere starren Körper. Neuentwickelt mit Grassmanns Punktrechnung. By A. Lotze. Leipzig, B. G. Teubner, 1922. 50 pp.

A great many quantities which enter into mechanics are vectors and consequently the most natural way to treat mechanics is by vector methods and this has been done by a great many writers. There are, however, difficulties in treating forces which act at a given point, for vectors in general are only determined in magnitude and direction and hence to locate them on definite lines brings in other considerations.

The Grassmann point analysis gives us, however, a natural way out of this difficulty. He considered two elements, $A-B$ (where A and B are points) which represents a vector in the ordinary sense of the word, and AB which represents the segment of the line joining the points A and B. In cases, then, when we wish to localize a vector we can indicate it by AB.

In this little pamphlet Lotze writes up quite an extensive treatment of mechanics from the point of view of Grassmann's analysis. He assumes a knowledge of the point analysis including the notions of the Lückenausdruck and the fraction. No discussion of this is given and in places the argument is not easy to follow. The author has introduced some symbols of his own or at least not known to the reviewer, e.g., in addition to Grassmann's complement he uses $\llcorner\ \bar{v}$ to indicate the vector, in a plane, into which $\bar{v}$ rotates by a positive rotation through $\pi/2$; $\perp\ \bar{v}$ indicates (in space) the 2-vector perpendicular to $\bar{v}$ and of equal magnitude and so directed that $\bar{v} \perp \bar{v} = v^2$. Different symbols are used to represent the quantities of different order and this lessens the difficulty of reading.

This is a fairly complete text of the mechanics of rigid bodies. It is divided into three chapters: I. Kinematics of rigid bodies; II. General dynamics of material point systems; III. Dynamics of rigid bodies. The general properties of rigid motion are quite fully treated in the first chapter. The second chapter carries us as far as the derivation of d'Alembert's and Hamilton's principles and Lagrange's equations. The last chapter deals with work and energy and the various screws such as the impulse screw and the force screw.

The pamphlet is well worth reading; but it seems to the reviewer as if the reading could have been made much easier.

C. L. E. MOORE

Précis d'Arithmétique. By J. Poirée. Paris, Gauthier-Villars et Cie., 1921. 62 pp.

C. Camichel has written a preface for this delightful little volume in which he says, "L'Arithmétique élémentaire est une excellente introduction à l'étude des Mathématiques. On y trouve sous une forme concrète des modèles de tous les modes de raisonnement depuis les plus simples jusqu'aux plus délicats de l'Analyse. Cependant cette partie des Mathématiques est en général négligée par les élèves." Poirée has presented a few topics from the theory of arithmetic and the theory of numbers in a way that will attract the neophyte and will be approved by the savant. The discussion commences with "Combien y a-t-il de billes?" and leads up to

"Recherche des Racines primitives d'un Nombre premier p." The idea of limit is introduced in connection with fractions, irrationals are connected with square root, logarithms with progressions. Classical theorems and concepts are introduced throughout the book: Euclid's proof of the infinitude of primes, sieve of Eratosthenes, number of divisors and sum of divisors of an integer, Cauchy's indicator, elementary theory of the congruence, Fermat's minor theorem, Wilson's theorem.

Possibly a book of this type needs no references, although some would probably lead a few readers to wider study. The book contains numerous good examples, but is without problems or stimulating questions for its reader. It is an engaging monograph with hardly a typographical flaw, and the reviewer believes that it will be of service either to organize and clarify the mathematical thinking of the younger or to direct students into number theory.

L. C. Mathewson

Latitude Developments Connected with Geodesy and Cartography. By Oscar S. Adams. Washington, United States Coast and Geodetic Survey, 1921. Special Publication No. 67. 132 pp.

Elements of Map Projection. By Charles H. Deetz and Oscar S. Adams. Washington, United States Coast and Geodetic Survey, 1921. Special Publication No. 68. 160 pp.

The first of these little books discusses the various "kinds of latitude" that arise in questions connected with geodesy and cartography. Five of these are discussed; namely, 1. *Geodetic, or astronomical, latitude* is the angle which the normal to the earth's surface at any point makes with the major axis of the meridian ellipse through the point; 2. *Geocentric latitude* is the angle which the radius vector makes with the major axis; 3. *Parametric latitude* is the parametric angle when the equation of the meridian ellipse is written in the usual parametric form; 4. *Isometric latitude* is connected with the conformal representation of the earth upon a sphere; 5. *Authalic latitude* is connected with the "equal area" representation of the earth upon a sphere (authalic from Tissot).

The book deals with formulas giving the last four in terms of the first and the eccentricity of a meridian. The formulas are applied to the Clarke spheroid of 1866, and there is a complete table for changing from one latitude to another.

The second book, as its title indicates, is devoted to various methods for constructing maps of the earth's surface, to a discussion of the advantages and disadvantages of each, and to the relative distortions introduced by each. The first half of the book is mostly descriptive and is geometrical in character. The latter half contains a full development of the Mercator projection with tables for computation, and also a mathematical discussion of various other projections. There is an explanation of the French "grid system" so much in use during the war.

The book has many excellent diagrams and maps and can be read by any one familiar with the calculus.

L. W. Dowling

Lessen over de Hoogere Algebra. Vol. I. By Fred. Schuh. Groningen, P. Noordhof, 1920. xx + 526 pp.

This work, which is to be complete in two volumes, was planned to be a ninth edition of Lobatto's *Higher Algebra* and on its title page is being announced as such. But although designed at its inception to be simply a revision of the eighth edition of Lobatto, the changes and additions which have arisen are so numerous as to warrant the appearance of these volumes as texts by Dr. Schuh himself. Very little has been taken bodily from Lobatto's text, the material taken over unchanged being primarily the examples and exercises used; but even with respect to these much new material has been added. These additions have so enlarged the work as to necessitate its publication in two volumes.

In volume I, the new material includes the treatment of theory of linear transformations and quadratic forms and the closely allied characteristic equation, called by Dr. Schuh the "secular equation" or "S-equation." To compensate for the quantity of additional material, there has been eliminated from the Lobatto text the treatment of permutations and combinations as they relate to the binomial theorem and to powers of polynomials, the reader being referred for these subjects to the secondary algebra published by Dr. Schuh in collaboration with P. Wydenes, which publication is to be considered as the connecting link between Wydenes' *Elementary Algebra* and the present two volumes of Dr. Schuh. Following the chapters dealing with determinants, linear equations and transformations, quadratic forms, simple properties of polynomials in x, are chapters in which D'Alembert's, Rolle's, Des Cartes' Budan-Fourier's and Sturm's theorems and their applications are treated in full. The concluding chapter treats the characteristic equation. The work is replete with examples, there being 815 in this first volume. The text is an admirable one suited to an advanced undergraduate course in college algebra.

J. N. VAN DER VRIES

Encyklopädie der Mathematischen Wissenschaften. Volume II, Part II. Edited by H. Burkhardt, W. Wirtinger, R. Fricke, and E. Hilb. Leipzig, B. G. Teubner, 1901–1921. xv + 897 pages.

This part of the German encyclopedia of mathematics contains articles by Osgood on the general theory of analytic functions of one and of several complex variables, by Wirtinger on algebraic functions and their integrals, by Fricke on elliptic functions and on automorphic functions including elliptic modular functions, by Hilb on linear differential equations in the complex domain and on non-linear differential equations, by Krazer and Wirtinger on Abelian functions and general theta functions. The first article was completed in 1901 and the last in 1920. The authors, editors, and publishers are to be congratulated on the completion of this part of the encyclopedia. It will be of the greatest service to the students of analysis. Indeed, the earlier articles have been in use for many years. Americans are proud that the article on analytic functions is written by a noted American mathematician who has contributed so much to the field.

A. D. PITCHER

Funktionentheoretische Vorlesungen. By H. Burkhardt. Berlin, Vereinigung Wissenschaftlicher Verleger. Band 1. Heft 1. *Algebraische Analysis.* Dritte umgearbeitete Auflage, besorgt von G. Faber, 1920. 10 + 182 pp. Band 1. Heft 2. *Einführung in die Theorie der Analytischen Funktionen einer Komplexen Veränderlichen.* Fünfte umgearbeitete Auflage, besorgt von G. Faber. 1921. 10 + 286 pp.

The first edition of this excellent and well known elementary treatment of function theory appeared in 1897. Since then it has appeared in three further editions, in 1903, 1907, and 1912 respectively, under the revision of the author. The latter died in 1914. This fifth edition of the second section and the third edition of the first section have been issued under the direction of Dr. G. Faber who succeeded Burkhardt at the Technische Hochschule in Munich. The first edition of Heft 2 was carefully reviewed by M. Bôcher in this BULLETIN, vol. 5 (1898–99), pp. 181–185. The second edition of Heft 2, together with the first edition of 1, was reviewed by L. E. Dickson in this BULLETIN, vol. 10 (1903–04), pp. 317–321. The third and fourth editions present no essential changes. It is to be noted that S. E. Rasor has provided a good English translation of the fourth edition and has supplied splendid sets of problems to accompany each chapter. The translation is published by D. C. Heath and Co.

In the present edition no fundamental changes appear, and the method and spirit of Burkhardt have been retained by Faber. Whatever slight changes have been made are in the interest of clarity and simplicity. These two books should continue to furnish an excellent introduction to the theory of functions, both real and complex.

H. J. ETTLINGER

Géométrie Perspective. By M. Emanaud. Paris, Octave Doin, 1921. xv + 432 pp.

The Encyclopédie Scientifique published under the direction of Dr. Toulouse contains as a sub-class the Bibliothèque de Mathématiques Appliquées edited by M. d'Ocagne of nomographic fame. The volume under review belongs to this series and has been written by M. Emanaud, the chief of graphic courses at the École polytechnique. In an introduction of 39 pages the author gives a brief account of the properties of collineations in two and three dimensions with special reference to their applications in perspective constructions. This is a very commendable feature of the book. In fact, a scientific base of this sort is indispensable for a successful and up-to-date course in perspective. When years ago the reviewer was teaching artistic perspective at the cantonal technicum in Biel, Switzerland, he followed the same method.

M. Emanaud's book is very ably written. It covers the usual topics in perspective. Of particular interest are the chapters on perspective in art, on theatrical perspective, and on the construction of dioramas and panoramas. The only serious criticism which the reviewer has to make pertains to the figures, which are very poorly drawn and lettered.

A. EMCH

NOTES

At the meeting of the Chicago Section of this Society on April 13–14 in Chicago, Professor S. Lefschetz, of the University of Kansas, will deliver a symposium lecture on *Curves traced on algebraic surfaces.*

The opening number of volume 23 of the TRANSACTIONS OF THIS SOCIETY contains the following papers: *Relatively uniform convergence and the classification of functions*, by E. W. Chittenden; *Periodic functions with a multiplication theorem*, by J. F. Ritt; *Note on Dirichlet and factorial series*, by Tomlinson Fort; *Functions of infinitely many variables in Hilbert space*, by W. L. Hart; *Prime and composite polynomials*, by J. F. Ritt; *Some two-dimensional loci connected with cross ratios*, by J. L. Walsh; *On transformations with invariant points*, by J. W. Alexander; *Invariant points in function space*, by G. D. Birkhoff and O. D. Kellogg. Volume 23 is the extra volume presented to the Society by an anonymous donor (see this BULLETIN, vol. 28, p. 234).

The opening number of volume 44 (1922) of the AMERICAN JOURNAL OF MATHEMATICS contains: *An arithmetical dual of Kummer's quartic surface*, by E. T. Bell; *Incidences of straight lines and plane algebraic curves and surfaces generated by them*, by Arnold Emch; *On the theorems of Gauss and Green*, by V. C. Poor; *An extension of the Sturm-Liouville expansion*, by C. C. Camp; *Conformal transformations of period n and groups generated by them*, by Harry Langman.

The December, 1921, number (vol. 23, No. 2) of the ANNALS OF MATHEMATICS contains: *Transformations of trajectories on a surface*, by Joseph Lipka; *On the structure of finite continuous groups with one two-parameter invariant subgroup*, by S. D. Zeldin; *On the simplification of the structure of finite continuous groups with more than one two-parameter invariant subgroup*, by S. D. Zeldin; *The automorphic transformation of a bilinear form*, by J. H. M. Wedderburn; *A direct determination of the minimum area between a curve and its caustic*, by Otto Dunkel; *The Poisson integral and an analytic function on its circle of convergence*, by A. Arwin; *Systems of circuits on two-dimensional manifolds*, by H. R. Brahana; *Two generalizations of the Stieltjes integral*, by P. J. Daniell.

Two volumes of the PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE DE L'UNIVERSITÉ DE STRASBOURG have appeared, in 1920 and 1922. They contain memoirs published by the members of the faculty of the University.

At the meeting of the National Academy of Sciences held in New York City November 14–16, 1922, the following mathematical papers were read: *On the attraction of a central body in the theory of relativity*, by G. D. Birkhoff; *The equiaffine geometry of paths*, by Oswald Veblen; *The cosmological equation of gravitation*, by Edward Kasner; *The use of isophelimatic lines in* *historical geography*, *A new theory of population*, *A new hydrodynamica phenomenon*, and *Further results in interior ballistics*, by A. G. Webster *Affine geometry of paths possessing an invariant integral*, by L. P. Eisenhart. The Draper Gold Medal was presented to Professor H. N. Russell, of Princeton University.

At the meeting of the International Research Council at Brussels, in July, 1922, the following executive committee was elected to serve until 1925: E. Picard, president; G. Lecointe and V. Volterra, vice-presidents; Sir A. Schuster, secretary general.

The following awards of Nobel prizes in physics have been announced: for 1921, to Professor Albert Einstein, for his theory of relativity and his general work in physics; for 1922, to Professor Neils Bohr, for his researches in the structure of atoms and radiation.

The Paris Academy of Sciences has awarded its Lalande Medal to Professor H. N. Russell, of Princeton University, and its Janssen Medal to Dr. Carl Stormer, professor of pure mathematics at the University of Christiania, for his work on the aurora borealis.

Dr. J. G. Hagen, director of the observatory of the Vatican, has been elected non-resident member of the Accademia dei Lincei.

The Royal Society of London has awarded its Sylvester Medal to Professor Tullio Levi-Civita, for his researches in geometry and mechanics.

The Franklin Medal of the Franklin Institute of Philadelphia has been presented to Sir J. J. Thomson.

Professor J. H. Jeans received the doctorate of science from Oxford University on the occasion of his delivery of the Halley lecture.

Professor N. E. Nörlund, of the University of Lund, has been called to the University of Copenhagen.

Professor F. von Dalwigk, of the University of Marburg, has been called to the Potsdam Geodetic Institute.

Dr. H. König, of the University of Göttingen, has been appointed professor of mathematics at the school of mines at Clausthal.

Dr. E. Kruppa has been appointed associate professor of mathematics at the Vienna Technical School.

Professor H. von Sanden, of the school of mines at Clausthal, has been appointed to a professorship at the Hannover Technical School.

Dr. C. Siegel, of the University of Göttingen, has been appointed professor of mathematics at the University of Frankfurt am Main.

Professor R. Mehmke, of the Stuttgart Technical School, has retired from active teaching.

Dr. A. Ostrowski has been admitted as privat docent at the University of Hamburg.

Dr. Janet has been appointed professor of pure mathematics at the University of Rennes.

Professor S. K. Banerji, of the department of applied mathematics at the University of Calcutta, has resigned to become director of the observatories at Bombay.

The Council of the University of Sheffield has appointed Professor A. H. Leahy emeritus professor of mathematics.

Mr. G. P. Thomson, lecturer in mathematics at Corpus Christi College, Cambridge, has been appointed professor of natural philosophy at the University of Aberdeen, as successor to Professor C. Niven, retired.

Mr. E. C. Francis, of Trinity College, has been elected fellow and lecturer in mathematics at Peterhouse, Cambridge.

Associate Professor C. F. Gummer has been promoted to a full professorship at Queen's University, Kingston.

Professor A. Sommerfeld will deliver a course of lectures at the Bureau of Standards, Washington, early in March, 1923, on the quantum theory and related subjects.

Associate Professor C. M. Sparrow, of the University of Virginia, has been promoted to a full professorship of physics.

Dr. T. H. Brown, of Yale University, has been appointed assistant professor of foreign trade at Columbia University.

Professor Glenn James, of the Carnegie Institute of Technology, has been appointed assistant professor of mathematics at the southern branch of the University of California.

Mr. R. M. Deming, of the Case School of Applied Science, has been appointed professor of mathematics at Upper Iowa University, Fayette.

Mr. F. H. Murray has been appointed instructor in mathematics at the University of West Virginia.

The death is reported of Professor A. V. Bäcklund, of Lund, at the age of seventy-seven years.

Professor C. A. Fischer, of Trinity College, Hartford, died December 7, 1922.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

Andersen (A. F.). Studier over Cesàro's summabilitetsmetode. Med særlig anvendelse overfor potensrækkernes teori. København, Gjellerup, 1921. 8vo. 100 pp.

Antoine (L.). 1: Sur l'homéomorphie de deux figures et de leurs voisinages. 2: Propositions données par la faculté. Thèses présentés à la faculté des sciences de Strasbourg. Paris, Gauthier-Villars, 1921. 4to. 105 pp.

Archimedes. Ueber Spiralen. Uebersetzt und mit Anmerkungen und einem Anhang versehen von A. Czwalina-Allenstein. (Ostwald's Klassiker der exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1922. 8vo. 71 pp.

Borel (E.). Méthodes et problèmes de la théorie des fonctions. Paris, Gauthier-Villars, 1922. 8vo. 148 pp.

Brunschvicg (L.). L'expérience humaine et la causalité physique. Paris, Alcan, 1922. 16 + 625 pp.

Cesàro (G.). Géométrie et crystallographie. Tous les polyèdres symétriques, non-superposables ou superposables à leurs images, peuvent etre déduits d'un petit nombre de formes très simples, en n'employant que la droite comme élément de symétrie. Bruxelles, Hayez, 1922.

Courant (C.). See Hurwitz (A.).

Cullis (C. E.). Chapters on algebra. Being the first three chapters of Matrices and determinoids, volume 3. Cambridge, University Press, 1920. 8vo. 9 + 182 pp.

Czwalina-Allenstein (A.). See Archimedes.

Dendy (A.). Problems of modern science. A series of lectures delivered at King's College, University of London. By various authors. Edited by A. Dendy. London, Harrap, 1922. 237 pp.

de Echaguibel (E.). Principios de análisis matematico. El problema fundamental de análisis. Bilbao, Eléxpuru Hermanos, 1922. 8vo. 286 pp.

Euler (L.). Opera omnia. Series I: Opera mathematica. VIII: Introductio in analysin infinitorum. Tomus primus. Ediderunt Adolph Krazer et Ferdinand Rudio. Leipzig, Teubner, 1922. 12 + 392 pp.

Freund (M.). Ueber das Potential mehrfach belegter Flächen. (Diss., Halle-Wittenberg.) Halle, Buchdruckerei Hohmann, 1922. 39 pp.

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Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 24, 1923.

Abstracts must be in the hands of the Secretary of the Section not later than February 10.

THE SAN FRANCISCO SECTION, at STANFORD UNIVERSITY, April 7, 1923.

Abstracts must be in the hands of the Secretary of the Section, B. A. Bernstein, not later than March 24.

CHICAGO, April 13 and 14, 1923. See the note on p. 476.

Abstracts must be in the hands of the Secretary of the Chicago Section, Arnold Dresden, not later than March 23.

NEW YORK CITY, April 28, 1923.

Abstracts must be in the hands of the Secretary of the Society not later than April 14.

R. G. D. RICHARDSON,
Secretary of the Society.

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Any communication intended for this BULLETIN may be addressed to the American Mathematical Society, Prince and Lemon Streets, Lancaster, Pa., or may be sent to separate officials of the Society as follows:

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, 304 Hicks Ave., Columbia, Mo. Reviews should be sent to J. W. YOUNG, Dartmouth College, Hanover, N. H. Notes should be sent to W. A. HURWITZ, No. 8 White Hall, Ithaca, N. Y.

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